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MATHEMATICS 30

MODULES 1–6

Learning Facilitator's Manual

Distance Learning
Alberta EDUCATION

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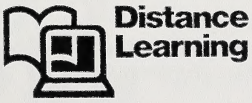
Mathematics 30

LEARNING FACILITATOR'S MANUAL



NOTE: This Mathematics 30 Learning Facilitator's Manual contains answers to teacher-assessed assignments and the final test; therefore, it should be kept secure by the teacher. Students should not have access to these assignments or the final test until they are assigned in a supervised situation. The answers should be stored securely by the teacher at all times.

This document is intended for	
Students	
Teachers (Mathematics 30)	✓
Administrators	
Parents	
General Public	
Other	

Mathematics 30
Learning Facilitator's Manual
Modules 1–6
Alberta Distance Learning Centre
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Teachers

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The Alberta Distance Learning Centre is dedicated to upgrading and continually improving your Learning Facilitator's Manual so that it accurately reflects any necessary revisions we have had to make in the Student Module Booklets, Assignment Booklets, or the sample final test. The types of revisions that will be made are those that make the course more accurate, current, or more effective.

The ADLC will send you the **latest enhancements** or **minor upgrades** for your Learning Facilitator's Manual if you return the following registration card to: Alberta Distance Learning Centre, Box 4000, Barrhead, Alberta T7N 1P4, Attention: Instructional Design and Development.

ADLC Learning Facilitator's Manual
Registration Card

First Name _____ Surname _____

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City _____ Postal Code _____

Course Title _____ Approximate Date of Purchase _____

You can help ensure that distance learning courseware is of top quality by letting us know of areas that need to be adjusted. Call the Alberta Distance Learning Centre free of charge by using the RITE line and ask for the Editing Unit. Also, a teacher questionnaire has been included at the back of most Learning Facilitator's Manuals. Please take a moment to fill this out.

We look forward to hearing from you!

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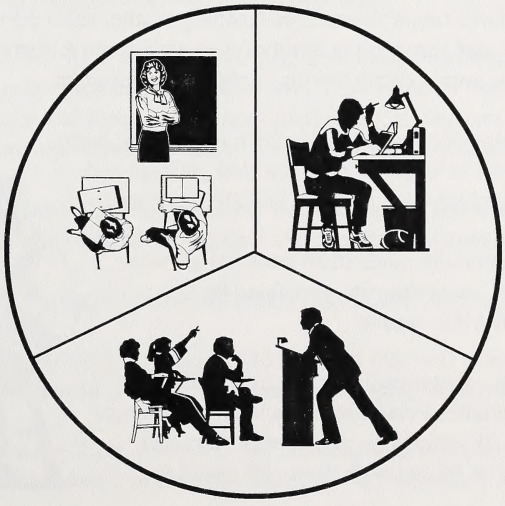
Introduction

A survey of these course materials will confirm that this new learning package has been specially designed for many kinds of teachers working in a variety of situations.

In Which Category Do You Fit?

- ❑ Small Schools Teacher
  - ❑ inexperienced
  - ❑ experienced, but in other subject areas
  - ❑ experienced in teaching Mathematics 30, but wanting to try a different approach
- ❑ Distance Learning Teacher
  - ❑ travelling to schools within the jurisdiction
  - ❑ using facsimile and teleconferences to teach students within the area
- ❑ Large Schools Teacher
  - ❑ inexperienced
  - ❑ experienced in teaching Mathematics 30, but wanting to try a different approach



Because these materials have been created by experienced classroom teachers and distance learning specialists, they have many advantages for students and teachers regardless of their situations.

Advantages for Students

- incorporates a strong learner-centred philosophy
- promotes such qualities in the learner as autonomy, independence, and flexibility
- is developed through media which suit the needs and circumstances of the learner
- reflects the experiential background of Alberta students
- opens up opportunities by overcoming barriers that result from geographical location
- promotes individualized learning, allowing learners to work at their own pace

Advantages for Teachers

- allows teachers maximum teaching time and minimizes preparation time
- includes different routes through the materials to suit different learners
- incorporates a wide range of teaching strategies, in particular those using independent and individual learning
- delivers curriculum designed by education specialists that reflects the Alberta Education Program of Studies with an emphasis on Canadian content
- provides learning materials which are upwardly compatible with advanced educational technology

Does it sound like something you could use?

This Learning Facilitator's Manual begins with an overview of the current Alberta Education Program of Studies for Mathematics 30. This summary is included for inexperienced teachers or those teachers who have found themselves teaching Mathematics 30 when their training is in other subject areas. This brief summary is not meant to replace the Alberta Education Program of Studies, but rather to help teachers confirm the highlights of the program.

Other parts of this introduction have also been included to help teachers become familiar with this new learning package and determine how they might want to use it in their classroom.

Beyond the introduction the guide itself contains answers, models, explanations, and other tips generated by the teachers who authored this course.

The module booklets, assignment booklets, and LFMs are the products of experienced classroom teachers and distance learning specialists. It is the hope of these teachers that their experience can be shared with those who want to take advantage of it.

Overview of the Program of Studies

To set goals and make informed choices, students need an array of thinking and problem-solving skills. Fundamental to this is an understanding of mathematical techniques and processes that will enable them to apply the basic skills necessary to address everyday mathematical situations, as well as acquire higher-order skills in logical analysis and methods for making valid inferences.

A knowledge of mathematics is essential for a well-educated citizenry. However, the need for and use of mathematics in the life of the average citizen is changing. Emphasis has shifted from the memorization of mathematical formulas and algorithms toward a more dynamic view of mathematics as a precise language, used to reason, interpret, and explore. There continues to be a need for the logical development of concepts and skills as a basis for the appropriate use of mathematical information to solve problems. Moreover, the use of available technology along with techniques such as estimation and simulation, incorporated with more traditional problem-solving techniques, are the tools by which mathematical problems are solved.

Change in the way in which mathematics is used is necessitating a concurrent change in the emphasis of a mathematics education. Students need an expanded list of fundamental concepts, but will also need to understand the ideas that make up those concepts and how they are related. They also require a familiarity with their applications. Most important, students have to be able to solve problems using the mathematical processes developed, and be confident in their ability to apply addition, the ability of technology to provide quick and accurate computation and manipulation, to enhance conceptual understanding, and to facilitate higher-order thinking, should be recognized and used by students.

The majority of students who enter high school exhibit mainly concrete operational behaviours with regard to mathematics. It is recognized that high school mathematics courses include many abstract understandings that students are expected to acquire. The course content of the high school mathematics program is cognitively appropriate for the students and should be presented in a way that is consistent with the students' ability to understand.

Upon conclusion of their high school mathematics education, students will be expected to be mathematically literate. Mathematical literacy refers to students' ability and inclination to manage the demands of their world through the use of mathematical concepts and procedures to communicate, reason, and solve problems. More specifically, *students will be expected to*

- have achieved understanding of the basic mathematical concepts, and developed the skills and attitudes needed to become responsible and contributing members of society
- apply basic mathematical skills and concepts in practical situations
- have developed the skills, concepts, and attitudes that will ensure success in the mathematical situations that occur in future educational endeavours, employment, and everyday life
- have developed the skills, concepts, and attitudes that will enable the acquisition of mathematical knowledge beyond the conclusion of secondary education

- have developed critical and creative thinking skills
- be able to communicate mathematical ideas effectively
- understand how mathematics can be used to investigate, interpret, and make decisions in human affairs
- understand how mathematics can be used in the analysis of natural phenomena
- understand the connections and interplay among various mathematical concepts and between mathematics and other disciplines
- understand and appreciate the positive contributions of mathematics, as a science and as an art, to civilization and culture

Students will be expected to demonstrate an attitude associated with mathematical literacy.

Following this are the **problem-solving** expectations that outline a variety of procedures, strategies, skills and checking techniques for solving problems. Because a major purpose for studying mathematics is to learn to solve problems, problem-solving expectations occur throughout all areas of the specific learner expectations. Students must have the background skills and knowledge necessary to achieve these expectations successfully, using problem-solving techniques.

The modules of the course are broken into a number of concepts. A **concept** is an abstract or general idea about specific instances that have common properties or an identifiable relationship to one another. The concepts are presented as mathematical definitions or theorems or as statements of mathematical ideas or abstractions. In the course of studies, concepts are stated as "*Students will be expected* to demonstrate an understanding that. . ."

Supporting each concept are a number of skills. **Skills** are intellectual or physical capabilities that will be developed within the context of the particular concept. In the course of studies, skills are stated as "*Students will be expected to. . .*"

Skills specifically related to the use of **technology** identify areas in which calculators and/or computer technology are applied by students as tools to be used for calculations, manipulation or graphing or to aid in the analysis of problems. Technology expectations are defined explicitly throughout the learner expectations. In many cases, a particular technology is indicated for investigation or analysis. It is in these situations that the use of technology enables students to engage in critical and creative thinking and problem solving.

A knowledge of mathematics is essential in many other areas of learning. With available information and the use of technology, the emphasis will not be so much on memorizing formulas but on how to use mathematical information to solve problems. Students will be expected to construct, investigate, translate, discuss, analyse, and formulate mathematics in a dynamic way.

The Mathematics 10–20–30–31 sequence is designed for students with an interest and aptitude in mathematics who are intending to pursue postsecondary studies at a university or in a mathematics-intensive program at a technical school or college. Having successfully completed Mathematics 30, students will have fulfilled the mathematics requirement for the Advanced High School Diploma.

Your mathematics course is divided into modules. Each module covers a major area of mathematical study, and is divided further into specific content areas called sections. Each module has its own Assignment Booklet which the student completes and submits at the end of that module.

Overview of Mathematics 30

Module	Title	Recommended Percentage of Available Instructional Time
Module 1	Polynomial Functions	15%
Module 2	Trigonometric and Circular Functions	18%
Module 3	Exponential and Logarithmic Functions	15%
Module 4	Sequences and Series	15%
Module 5	Quadratic Relations	15%
Module 6	Permutations, Combinations, and Statistics	22%

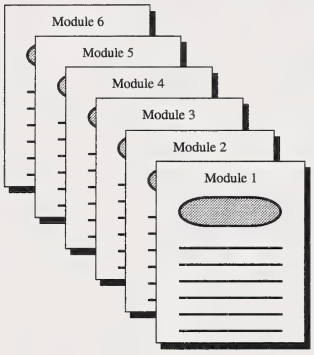
The Mathematics 30 course is the final course in the Mathematics 10–20–30 academic series. The course is divided into six modules as outlined. The course is offered for five credits only.

Structure of the Learning Package

Basic Design

This new learning package involves many other components in addition to the Learning Facilitator's Manual.

Modules



The print components involve many booklets called modules. These modules contain guided activities that instruct students in a relevant, realistic setting.

The modules have been specially designed to promote such qualities in the learner as autonomy, independence, and flexibility. Writers have incorporated such teaching strategies as working from the concrete to the abstract, linking the old to the new, getting students actively involved, and using advance, intermediate, and post organizers. Many other techniques enable learners to learn on their own for at least some of the time.

Contents
Overview Evaluation
Section 1 Activity 1 Activity 2 etc.
Section 2 Activity 1 Activity 2 etc.
Section 3 Activity 1 Activity 2 etc.
Section 4 Activity 1 Activity 2 etc.
Module Summary

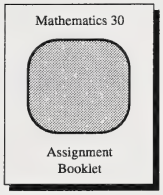
The structure of the module booklets follows a systematic design. Each module begins with a detailed table of contents which shows the students all the main steps. It acts as an organizer for students. The overview introduces the module topic or theme. A graphic representation has been included to help visual learners and poor readers. The introduction also states the weightings of each assignment.

The body of the module is made up of two or more closely related sections. Each section contains student activities that develop skills and knowledge centred around a theme.

The activities may involve print, audio, video, computer, or laser videodisc formats. At times the student and the learning facilitator are allowed to choose the activity that best suits the student's needs and interests. Other activities such as the Extra Help and Enrichment are optional pathways. This flexibility caters to each student's personal situation.

The summary focuses on the skills and strategies that the student has learned.

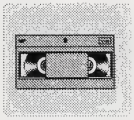
Assignment Booklet



Accompanying each module is an assignment booklet. The activities in these booklets can be used for formative and for summative assessments. The students should complete these assignment booklets when they have thoroughly reviewed the module materials. The assignment booklets have been designed for classroom use, or for mailing. **If the booklets are not being mailed, you should remove the outside cover.**

Media

COMPUTER DISK



VIDEOCASSETTE

The package also includes references to media. Some types of media such as computer disks are optional choices for students; however, there are activities that require students to view certain videos. These mandatory videos are listed on the following page. It is important that you acquire these videos as you are planning the course. In addition to the mandatory videos, optional videos have been mentioned at various points in the modules. A list of the optional videos is also included on the following page. More information about the videos can be found within the LFM.

Materials, Media, and Equipment

Mandatory Components

Equipment (Hardware)	Media	Materials
VCR	Mandatory Video List • Mathematics 30 Video 1 Graphing Mathematical Concepts: – Graphing Trigonometric Functions 1 • Mathematics 30 Video 3 Discovering Conics: – Cutting the Curves – Graphing the Curves – Defining the Locus – Eccentricity	• LFM for Mathematics 30 • one complete set of module booklets (6) and assignment booklets (6) for each student • There is a final test.

Videocassettes used in the course may be available from the Learning Resources Distributing Centre or ACCESS Network. You may also wish to call your regional library service for more information.

Optional Components

Equipment (Hardware)	Media	Materials
VCR	Optional Video List • Mathematics 30 Video 1 Graphing Mathematical Concepts: – Graphing Polynomial Functions – Graphing Logarithms and Exponents – Graphing Statistics • Mathematics 30 Video 2 Catch 30: – Arithmetic Sequences and Series – Geometric Sequences and Series – Permutations and Combinations – Statistics	

Using This Learning Package in the Classroom

Conventional Classroom

Whether your classroom has desks in rows or tables in small groups, you may be most comfortable with a learning system that you can use with all your students in a paced style. In other words, you may want a package that will suit all of your students, so they can move through the materials as one group or several small groups. Because these materials contain different routes or pathways within each module, they can address various learning styles and preferences. The materials also include many choices within the activities to cater to different thinking levels and ability levels. Because of their versatility and flexibility, these materials can easily suit a conventional classroom.

Open-Learning Classroom

Open learning is the concept of opening up opportunities by overcoming barriers of time, pace, and place by giving the learners a package specially designed to enable them to learn on their own for at least some of the time.

Such a concept is not new. Many teachers can recite attempts to establish an individualized learning system as they recognized the importance of trying to personalize courseware to meet each individual student's needs. But these efforts often failed due to lack of time and lack of quality materials that conformed to Alberta specifications.

Due to advanced educational technology and improved Alberta-specific learning packages, a student-centred approach is now possible. Improved technology now allows us to provide support to learners individually, regardless of their pace or location. A teacher cannot be in twenty-eight places at one time offering guidance. However, media and a well-designed learning package can satisfy individual needs. Technology can also help provide an effective management system needed to track the students as they progress independently through the materials.

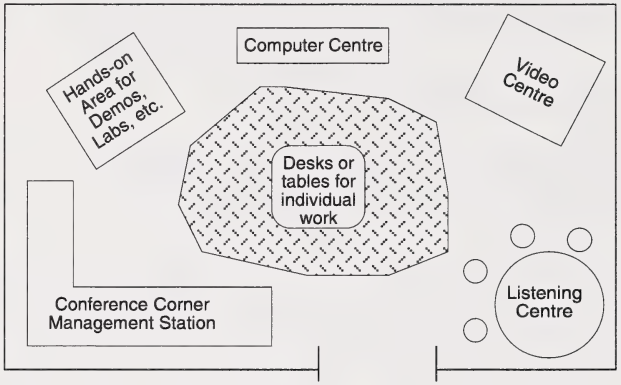
The key to a successful open-learning system depends on three vital elements: a learning package specially designed to enable students to learn effectively on their own for at least some of the time; various kinds of learner support; and a management system and style that ensures that the open-learning system runs smoothly.

The Key to a Successful Open-Learning System

Learning Package

The specially designed learning package needed for a successful open-learning system has been developed for you. The objectives teach current Alberta specifications using strategies designed for individualized instruction. As the learning facilitator, you need to be sure to have all the components in the learning package available to students as needed.

If adequate numbers of media are available to satisfy the demand, a centre can be established for specific media.



You may not have the luxury to have enough hardware to set up a permanent video or computer centre in your classroom. In that case, students should be encouraged to plan ahead. Perhaps every three to five days they should preview their materials and project when they would need a certain piece of media. This would allow you to group students, if necessary, or reserve media as required.

Support

Support is definitely a key element for successful learning, and when you're planning an individualized, non-paced program, you need to carefully plan when and how support will be given.

The materials contain a form of consistent support by providing immediate feedback for activities included in the module booklet. High school students have solutions, models, explanations, and guides included in the appendix of every module booklet. These are included so students can receive immediate feedback to clarify and reinforce their basic understanding before they move on to higher levels of thinking.

As the learning facilitator, you may be needed to offer more personal guidance to those students having difficulty, or you may need to reinforce the need for students to do these activities carefully before attempting the assignments in the assignment booklet.

The activities include choices and pathways. If a student is having difficulty, you may need to encourage that student to work on all the choices rather than one. This would provide additional instruction and practice in a variety of ways.

Another form of support is routine contact with each individual. This might be achieved with a biweekly conference scheduled by you, or as students reach a certain point (e.g., after each section is completed), they may be directed to come to the conference area.

Special counselling may be needed to help students through difficult stages. Praise and encouragement are important motivators, particularly for those students who are not used to working independently.

Direct teaching may be needed and scheduled at certain points in the program. This might involve small groups or a large group. It might be used to take advantage of something timely (e.g., election, eclipse, etc.), something prescheduled like the demonstration of a process, or something involving students in a hands-on, practical experience.

Support at a distance might include tutoring by phone, teleconferencing, faxing, or planned visits. These contacts are the lifeline between learners and distance education teachers, so a warm dialogue is essential.

Management

Good management of an open-learning system is essential to the success of the program. The following areas need action to ensure that the system runs smoothly:

- Scheduling, Distributing, and Managing Resources – As discussed earlier, this may require a need for centres or a system for students to project and reserve the necessary resources.

- Scheduling Students – Students and teachers should work together to establish goals, course completion timelines, and daily timelines. Although students may push to continue for long periods of time (e.g., all morning), teachers should discourage this. Concentration, retention, and motivation are improved by taking scheduled breaks.

- Monitoring Student Progress – You will need to record when modules are completed by each student. Your data might also include the projected date of completion if you are using a student contract approach.

Sample of a Student Progress Chart

Mathematics 30		Module 1	Module 2	Module 3	Module 4	Module 5	Module 6	Final Test
Billy Adams	P							
	A							
Louise Despins	P							
	A							
Violet Klaissian	P							
	A							
P = Projected Completion Date A = Actual Completion Date								

The student could keep a personal log as well. Such tracking of data could be stored easily on a computer.

- Recording Student Assessments – You will need to record the marks awarded to each student for work completed in each module assignment booklet. The marks from these assignment booklets will contribute to a portion of the student's final mark. Other criteria may also be added (a special project, effort, attitude, etc.). Whatever the criteria, they should be made clear to all students at the beginning.

Sample of a Student Assessment Chart

Mathematics 30	Module 1	Module 2	Module 3	Module 4	Module 5	Module 6	Year's Average	Final Test	Final Mark
Billy Adams	**67**	**65**	**54**	**47**	**78**	**67**	**63**		
Louise Despins	**43**	**50**	**54**	**55**	**48**	**42**	**49**		
Violet Klaissian	**65**	**65**	**66**	**68**	**67**	**70**	**67**		

Letter grading could easily be substituted.

- Recording Effectiveness of System – Keep ongoing records of how the system is working. This will help you in future planning.

Sample of a System Assessment Chart

Module 1			
Date	Module Booklet	Assignment Booklet	Resources/Media

The Role of the Teacher in an Open-Learning Classroom

The teachers in a conventional classroom spend a lot of time talking to large groups of learners. The situation in open learning requires a different emphasis. Teachers will probably meet learners individually or in very small groups.

With this approach it is necessary to move beyond the idea of a passive learner depending largely on a continually supportive teacher. The teacher must aim to build the student's confidence, to stimulate the learner into self-reliance, and to guide the learner to take advantage of routes that are most meaningful and applicable to the learner.

These materials are student-centred, not teacher-centred. The teacher needs to facilitate learning by providing general support to the learner.

Evaluation

Evaluation is important to the development of every learner. Data gathering and processing, and decision making, at the student and teacher level, serve as means of identifying strengths and weaknesses.

These specially designed learning packages contain many kinds of informal and formal evaluation.

Observation

In the classroom the teacher has the opportunity to see each student perform every day and to become aware of the level and nature of each student's performance.

Observations are more useful if they are recorded in an organized system. The following list of questions is a sample of types of observations and how they can be collected.

Observation Checklist

	B. Adams	L. Despins	V. Klaissian	H. Smith	K. Dalley
1. Does the student approach the work in a positive manner?					
2. Is the student struggling with the reading level?					
3. Does the student make good use of time?					
4. Does the student apply an appropriate study method?					
5. Can the student use references effectively, etc.?					

Observation may suggest a need for an individual interview with a student.

Individual Conferences

Individual conferences may be paced (scheduled) by the calendar, at certain points in the module, or they may be set up only as needed or requested.

During these conferences teachers can determine the student's progress and can assess the student's attitudes toward the subject, the program, school, and self, as well as the student's relationship with other students. With guided questions the teacher can encourage oral self-assessment; the student can discuss personal strengths or weaknesses in regard to the particular section, module, or subject area.

Self-Appraisal

Self-appraisal helps students recognize their own strengths and weaknesses. Through activities that require self-assessment, students also gain immediate feedback and clarification at early stages in the learning process. Teachers need to promote a responsible attitude toward these self-assessment activities. Becoming effective self-assessors is a crucial part of becoming autonomous learners. By instructing, motivating, providing positive reinforcement, and systematically supervising, the learning facilitator will help students develop a positive attitude toward their own progress.

For variation, students may be paired and peer-assessing may become part of the system. The teacher may decide to have the student self-assess some of the activities, have a peer assess other activities, and become directly involved in assessing the remainder of the activities.

When the activities have been assessed, the student should be directed to make corrections. This should be made clear to students right from the start. It is important to note the correct association between the question and the response to clarify understanding, aid retention, and be of use for study purposes.

Many of the activities include choices for the student. If the student is having difficulty, more practice may be warranted, and the student may need to be encouraged to do more of the choices.

Each section within a module includes additional types of activities called Extra Help and Enrichment. Students are expected to be involved in the decision as to which pathway best suits their needs. They may decide to do both.

Self-appraisal techniques can also be introduced at the individual conferences. Such questions as the following might be included:

- What steps are you taking to improve your understanding of this topic?
- What method of study do you use most?
- How do you organize your material to remember it?
- What steps do you follow when doing an assignment?
- What could you do to become an even better reader?
- Do you have trouble following directions?
- Did you enjoy this module?

A chart or checklist could be used for recording responses.

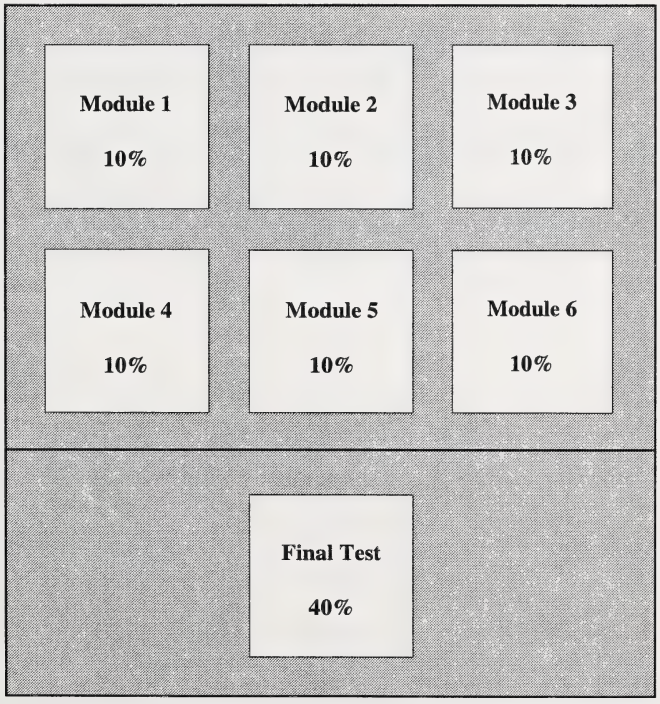
Informal Evaluation: Assignments

Informal evaluation, such as the assignments included in each module, are an invaluable aid to the teacher. They offer ongoing assessment information about the student's achievement and the behaviour and attitudes that affect that achievement.

Each module contains a separate booklet called the Assignment Booklet. This booklet assesses the knowledge or skills that the student has gained from the module. **The student's mark for the module may be based solely on the outcome of learning evident in the assignment booklet; however, you may decide to establish a value for other variables such as attitude or effort.** It is important that you establish at the beginning which outcomes will be evaluated, and that all students clearly understand what is expected.

Final Test

All LFMs include a formal final test which can be photocopied for each member of the class. The test, closely linked to the learning outcomes stated in the module booklets, gives the teacher precise information concerning what each student can or cannot do. Answers, explanations, and marking guides are also included. The value of the final test and each module is the decision of the classroom teacher. Following is a suggestion only.



Introducing Students to the System

Your initiation to these learning materials began with a basic survey of what was included and how the components varied. This same process should be used with the class. After the materials have been explored, a discussion might include the advantages and the disadvantages of learning independently or in small groups. The roles of the students and teacher should be analysed. The necessary progress checks and rules need to be addressed. Your introduction should motivate students and build a responsible attitude toward learning autonomously.

Skill Level

It is important for students to understand that there are certain skills that they will need in order to deal successfully with the course materials. They are listed below:

- understanding and using instructional materials (table of contents, index, list of illustrations, appendices, bibliography, and glossary)
- interpreting maps, graphs, and charts
- using reference materials
- recognizing special symbols
- using a scientific calculator and a graphing calculator

Other general skills are using reliable study methods, outlining, and learning to read at a flexible rate.

To decide the level and amount of instruction needed to accommodate the varied levels among students, you may wish to prepare and administer skill inventories or pretests. If most students need help with a particular skill, you may want to plan a total class instructional session. If only certain students lack a skill, you may want to set up a temporary skill group to help students who need it, or you may want to develop a skills file for this purpose.

Reading Level

These course materials are largely print based, but poorer readers need not be discouraged. It is important that you assure the students that these materials have been designed for easy reading. The authors have employed special strategies that lower and control the reading level. Some of them are

- the conscious selection of vocabulary and careful structuring of sentences to keep the materials at an independent reading level
- the integration of activities, examples, and illustrations to break text into appropriate-sized chunks
- the inclusion of many kinds of organizers (advance, graphic, intermediate, concept mapping, post organizers) to help give students a structure for incorporating new concepts

- the recognition that vocabulary and concepts are basic to understanding content materials and, thus, must be handled systematically (defined in context, marginal notes, footnotes, and often in a specialized glossary)

- the acknowledgement that background knowledge and experience play a vital role in comprehension

- the systematic inclusion of illustrations and videos to help poorer readers and visual learners, and audiocassettes and software as an alternative to print-based learning

- a variety of formats (paragraphs, lists, charts, etc.) to help poorer readers who do not absorb or retain main ideas easily in paragraph format

- the inclusion of media and activity choices to encourage an active rather than passive approach

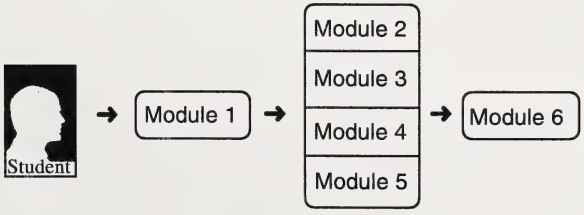
- instruction in a meaningful setting rather than in a contrived, workbook style

- using purposeful reading, viewing, and doing to produce better interpretation of the course materials

- the recognition that students need structured experiences when reading, viewing, or listening to instructional materials: developing pupil readiness, determining the purpose, providing guided instruction and feedback, rereading if necessary, and extending (This structure closely resembles the reading process.)

To help make the learning package more readable, you can begin your module preparation by reading (viewing, listening to) all the related materials that are going to be used. You need a solid background in order to assess and develop a background knowledge for students. The students' experiential bases may be assessed through brainstorming sessions concerning the topic, or by using visuals and guided questions to predict what the topic might be about.

It is recommended that you start with Module 1 because this module includes basic introductory information, and it is also recommended that you end with Module 6 because this module acts as a summary or culmination.



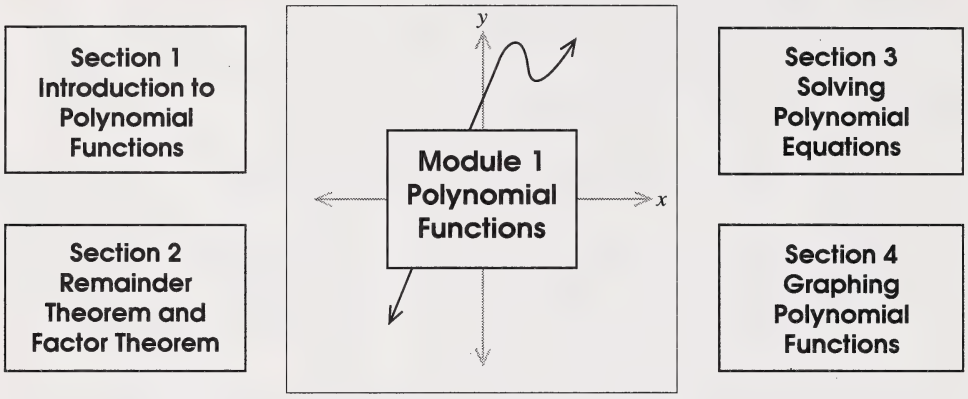
Module 1: Polynomial Functions

Overview

The study of polynomial functions emphasizes conceptual understanding, coming to understand algebra as a means of representing general cases and a problem-solving tool. Students interpret the graphs of functions, explore the properties of graphs, and determine how these properties relate to the forms of the corresponding equations.

Polynomial functions are very useful for describing relations among variables in a vast array of real-world situations.

The analysis of polynomial functions is a very important foundation for the study of calculus.



Evaluation

The evaluation of this module is based on four assignments and a final module assignment:

Section 1 Assignment	**8 marks**
Section 2 Assignment	**18 marks**
Section 3 Assignment	**26 marks**
Section 4 Assignment	**18 marks**
Final Module Assignment	**30 marks**
TOTAL	**100 marks**

Optional Video

Section 4: Extra Help

Graphing Polynomial Functions, *Graphing Mathematical Concepts* series, ACCESS Network, available from Learning Resources Distributing Centre.

Section 1: Introduction to Polynomial Functions

Key Concepts

- recognizing a polynomial function
- classifying a given polynomial function according to degree and coefficients (integral, rational, or real)
- dividing an integral polynomial in one variable by a binomial of the form $x-a$ using the division algorithm
- dividing an integral polynomial in one variable by a binomial of the form $x-a$ using synthetic division

Section 1: Assignment Answer Key (8 marks)

1. Yes, the polynomial is a function. **(1 mark)**

2. **a.** The term with the highest degree in a polynomial determines the degree of the polynomial. The degree of the polynomial is 4 since the term $-3x^4$ has a degree of 4. **(1 mark)**

b. This is a real polynomial since the coefficients 2, -3, and $\sqrt{7}$ belong to the real number system. The term $\sqrt{7}$ is irrational and it is a member of the real number system. **(1 mark)**

3.
$$\begin{array}{r|l} & 4x^2+3x+6 \\ \hline x-2 & 4x^3-5x^2+0x+3 \\ & \underline{4x^3-8x^2} \\ & \quad 3x^2+0x \\ & \quad \underline{3x^2-6x} \\ & \qquad 6x+3 \\ & \qquad \underline{6x-12} \\ & \qquad\quad 15 \end{array}$$

$Q(x)=4x^2+3x+6$ and $R=15$. **(2 marks)**

4. **a.** **Method 1: Subtraction Method**

Since $2x-3=2\left(x-\frac{3}{2}\right)$, use $-\frac{3}{2}$ for the divisor.

$$\begin{array}{r|cccc} -\frac{3}{2} & 6 & -5 & 0 & -9 \\ & & -9 & -6 & -9 \\ \hline & 6 & 4 & 6 & \boxed{0} \end{array}$$

To obtain the quotient, divide these coefficients by 2. Thus, $Q(x)=3x^2+2x+3$ and $R=0$.

Method 2: Addition Method

Since $2x-3=2\left(x-\frac{3}{2}\right)$, use $\frac{3}{2}$ for the divisor.

$$\begin{array}{c|cccc} \frac{3}{2} & 6 & -5 & 0 & -9 \\ & & 9 & 6 & 9 \\ \hline & 6 & 4 & 6 & \boxed{0} \end{array}$$

To obtain the quotient, divide these coefficients by 2. Thus, $Q(x)=3x^2+2x+3$ and $R=0$.

(2 marks)

b. Since $R=0$, $\left(x-\frac{3}{2}\right)$ is a factor of $6x^3-5x^2-9$.

$$\begin{aligned} P(x)&=\left(x-\frac{3}{2}\right)\left(6x^2+4x+6\right) \\ &=\left(x-\frac{3}{2}\right)(2)\left(3x^2+2x+3\right) \\ &=(2x-3)\left(3x^2+2x+3\right) \quad \textbf{(1 mark)} \end{aligned}$$

Section 2: Remainder Theorem and Factor Theorem

Key Concepts

- evaluating an integral polynomial function for a given value of the domain by substitution
- developing and prove the Remainder Theorem
- evaluating an integral polynomial function for a given value of the domain using the Remainder Theorem
- developing and prove the Factor Theorem

Section 2: Assignment Answer Key (18 marks)

1. a. $$\begin{aligned} P(-1)&=3(-1)^3-8(-1)^2+2(-1)-5 \\ &=-3-8-2-5 \\ &=-18 \quad \textbf{(1 mark)} \end{aligned}$$

b. $$\begin{aligned} P\left(\sqrt{2}\right)&=3\left(\sqrt{2}\right)^3-8\left(\sqrt{2}\right)^2+2\left(\sqrt{2}\right)-5 \\ &=3\left(2\sqrt{2}\right)-8(2)+2\sqrt{2}-5 \\ &=6\sqrt{2}-16+2\sqrt{2}-5 \\ &=8\sqrt{2}-21 \quad \textbf{(1 mark)} \end{aligned}$$

2. If a polynomial $P(x)$ is divided by $x-a$, the remainder R is equal to $P(a)$.

$$\begin{aligned} P(3)&=-2(3)^3+2(3)^2+7(3)-5 \\ &=-54+18+21-5 \\ &=-20 \quad \textbf{(2 marks)} \end{aligned}$$

3. $P(-2)=(-2)^4-d(-2)^3+5(-2)+6$

$52=16+8d-10+6$

$52=12+8d$

$40=8d$

$d=5$ **(2 marks)**

4. A polynomial $P(x)$ has the binomial $x-a$ as a factor if $P(a)=0$.

$$\begin{aligned}P(-3)&=2(-3)^3+5(-3)^2-4(-3)-3\\&=-54+45+12-3\\&=0\end{aligned}$$

Therefore, $x+3$ is a factor.

Method 1: Long Division

$$\begin{array}{r}2x^2-1x-1\\ x+3\overline{)\,2x^3+5x^2-4x-3}\\ \underline{2x^3+6x^2}\\ -1x^2-4x\\ \underline{-1x^2-3x}\\ -1x-3\\ \underline{-1x-3}\\ 0\end{array}$$

Therefore, $a=2$, $b=-1$, and $c=-1$.
(3 marks)

Method 2: Synthetic Division

$$\begin{array}{r|rrrr}3 & 2 & 5 & -4 & -3\\ & & 6 & -3 & -3\\ \hline & 2 & -1 & -1 & \boxed{0}\end{array}$$

$Q(x)=2x^2-x-1$

Therefore, $a=2$, $b=-1$, and $c=-1$.

5. The binomial $x-2$ is a factor if $P(2)=0$.

$$\begin{aligned}P(2)&=2(2)^4-5(2)^3+3(2)^2-4(2)+4\\&=32-40+12-8+4\\&=0\end{aligned}$$

Thus, $x-2$ is a factor.

The number a is a zero of a polynomial $P(x)$ if $P(a)=0$.
Since $P(a)=0$, then 2 is a zero of the polynomial. **(2 marks)**

6. If a polynomial $P(x)$ is divided by $x-a$, the remainder R is equal to $P(a)$.

If the divisor is $(x-1)$, then $P(1)=-187$.

$$P(1)=3(1)^3-d(1)+c$$
$$-187=3-d+c$$
$$-d+c=-190 \quad (1)$$

If the divisor is $(x+4)$, then $P(-4)=-2$.

$$P(-4)=3(-4)^3-d(-4)+c$$
$$-2=-192+4d+c$$
$$4d+c=190 \quad (2)$$

$$-d+c=-190 \quad (1)$$
$$4d+c=190 \quad (2)$$
$$(1)-(2): \quad -5d=-380$$
$$d=76$$

Substitute $d=76$ into (1).

$$-76+c=-190$$
$$c=-114$$

The value of d is 76 and the value of c is -114. **(3 marks)**

7. a. If $x+2$ is a factor, then $P(-2)=0$.

$$P(-2)=3(-2)^3-2(-2)^2-19(-2)-6$$
$$=-24-8+38-6$$
$$=0$$

Thus, $x+2$ is a factor. **(1 mark)**

b. Method 1: Long Division

$$\begin{array}{r} 3x^2-8x-3 \\ x+2\overline{)\,3x^3-2x^2-19x-6} \\ \underline{3x^3+6x^2} \\ -8x^2-19x \\ \underline{-8x^2-16x} \\ -3x-6 \\ \underline{-3x-6} \\ 0 \end{array}$$

$$P(x)=(x+2)\left(3x^2-8x-3\right)$$
$$=(x+2)(3x+1)(x-3) \quad \textbf{(2 marks)}$$

Method 2: Synthetic Division

$$\begin{array}{r|rrrr} 2 & 3 & -2 & -19 & -6 \\ & & 6 & -16 & -6 \\ \hline & 3 & -8 & -3 & \boxed{0} \end{array}$$

$$Q(x)=3x^2-8x-3$$
$$=(3x+1)(x-3)$$

$$P(x)=(x+2)(3x+1)(x-3)$$

c. The zeros are -2, $-\frac{1}{3}$, and 3. **(1 mark)**

Section 3: Solving Polynomial Equations

Key Concepts

- developing the Integral Zero Theorem and the Rational Zero Theorem
- factoring a given integral polynomial over the integers
- applying the Factor Theorem to solve equations involving polynomials of degree greater than two
- determining an integral polynomial function given its rational zeros

Section 3: Assignment Answer Key (26 marks)

1. The potential integral zeros are factors of the constant term -16. The potential integral zeros are ± 1, ± 2, ± 4, ± 8, and ± 16. **(1 mark)**

2. The potential integral zeros are ± 1 and ± 3, since these are factors of the constant term -3.

If $P(a) = 0$, then a is a zero.

$P(1) = 1^3 + 3(1)^2 - 1 - 3$
$= 1 + 3 - 1 - 3$
$= 0$

Thus, 1 is a zero.

$P(-1) = -1 + 3 + 1 - 3$
$= 0$

Therefore, -1 is a zero.

$P(3) = 27 + 27 - 3 - 3$
$\neq 0$

Thus, 3 is not a zero.

$P(-3) = -27 + 27 + 3 - 3$
$= 0$

Therefore, -3 is a zero.

The integral zeros are 1, -1, and -3. **(2 marks)**

3. The potential numerators are factors of the constant term -8. The potential denominators are factors of the coefficient of the highest degree term. This coefficient is 3. The potential rational zeros are all possible combinations of these factors when the potential numerators are divided by the potential denominators.

The potential numerators are ± 1, ± 2, ± 4, and ± 8.

The potential denominators are ± 1 and ± 3.

The potential rational zeros are ± 1, ± 2, ± 4, ± 8, $\pm\frac{1}{3}$, $\pm\frac{2}{3}$, $\pm\frac{4}{3}$, and $\pm\frac{8}{3}$. **(1 mark)**

4. The potential numerators are factors of the constant term 4. The potential numerators are ± 1, ± 2, and ± 4.

The potential denominators are factors of the coefficient of the highest degree term. This coefficient is 3. The potential denominators are ± 1 and ± 3.

The potential rational zeros are all possible combinations of these factors when the potential numerators are divided by the potential denominators. The potential rational zeros are ± 1, ± 2, ± 4, $\pm\frac{1}{3}$, $\pm\frac{2}{3}$, and $\pm\frac{4}{3}$.

$$P(1)=3(1)^3+(1)^2-8(1)+4$$
$$=3+1-8+4$$
$$=0$$

Therefore, 1 is a zero.

Use -1 as the divisor in the synthetic division method.

$$\begin{array}{r|rrrr} -1 & 3 & 1 & -8 & 4 \\ & & -3 & -4 & 4 \\ \hline & 3 & 4 & -4 & \boxed{0} \end{array}$$

Since the remainder is 0, then $x-1$ is a factor, and 1 is a zero. This checks with direct substitution.

$$P(x)=(x-1)\left(3x^2+4x-4\right)$$

Now factor $3x^2+4x-4$.

$$3x^2+4x-4=(3x-2)(x+2)$$

Let $(3x-2)(x+2)=0$.

$(3x-2)=0$ or $x+2=0$

$3x=2$ $\qquad$ $x=-2$

$x=\frac{2}{3}$

The rational zeros are 1, $\frac{2}{3}$, and -2. **(3 marks)**

5. $P(1)=(1)^4-7(1)^3+15(1)^2-11(1)+2$
$=1-7+15-11+2$
$=0$

Therefore, 1 is a zero.

Use -1 as the divisor in the synthetic division method (subtraction method).

$$\begin{array}{r|rrrr} -1 & 1 & -7 & 15 & -11 & 2 \\ & & -1 & 6 & -9 & 2 \\ \hline & 1 & -6 & 9 & -2 & \boxed{0} \end{array}$$

Since the remainder is 0, then $x-1$ is a factor and 1 is a zero.

$$P(x)=(x-1)\left(x^3-6x^2+9x-2\right)$$

Now apply the synthetic division method to the coefficients $1, -6, 9,$ and -2.

$$\begin{array}{r|rrrr} 1 & 1 & -6 & 9 & -2 \\ & & 1 & -7 & 16 \\ \hline & 1 & -7 & 16 & \boxed{-18} \end{array}$$

Since the remainder is not 0, then -1 is not a zero.

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 9 & -2 \\ & & 2 & -16 & 50 \\ \hline & 1 & -8 & 25 & \boxed{-52} \end{array}$$

Since the remainder is not 0, then -2 is not a zero.

$$\begin{array}{r|rrrr} -2 & 1 & -6 & 9 & -2 \\ & & -2 & 8 & -2 \\ \hline & 1 & -4 & 1 & \boxed{0} \end{array}$$

Since the remainder is 0, then $x-2$ is a factor and 2 is a zero.

$$P(x)=(x-1)(x-2)\left(x^2-4x+1\right)$$

Since x^2-4x+1 cannot be factored any further, the factored form of $P(x)$ over the integers is $P(x)=(x-1)(x-2)\left(x^2-4x+1\right)$. The zeros of x^2-4x+1 can be determined by using the quadratic formula.

$$\begin{aligned} x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &= \frac{-(-4)\pm\sqrt{(-4)^2-4(1)(1)}}{2(1)} \\ &= \frac{4\pm\sqrt{12}}{2} \\ &= \frac{4\pm2\sqrt{3}}{2} \\ &= 2\pm\sqrt{3} \end{aligned}$$

The real zeros are $1, 2, 2+\sqrt{3}$, and $2-\sqrt{3}$. **(5 marks)**

6. Solving this equation for its roots is similar to determining the zeros of $P(x)=12x^3+17x^2+2x-3$.

$$\begin{aligned} P(1) &= 12(1)^3+17(1)^2+2(1)-3 \\ &= 12+17+2-3 \\ &\neq 0 \end{aligned}$$

Thus, 1 is not a root.

$$\begin{aligned} P(-1) &= 12(-1)^3+17(-1)^2+2(-1)-3 \\ &= -12+17-2-3 \\ &= 0 \end{aligned}$$

Thus, -1 is a root.

Use 1 as the divisor in the synthetic division method (subtraction method).

$$\begin{array}{c|cccc} 1 & 12 & 17 & 2 & -3 \\ & & 12 & 5 & -3 \\ \hline & 12 & 5 & -3 & \boxed{0} \end{array}$$

Since the remainder is 0, then $x+1$ is a factor, and -1 is a zero.

The factored form is $(x+1)\left(12x^2+5x-3\right)=0$.

Now factor $12x^2+5x-3$.

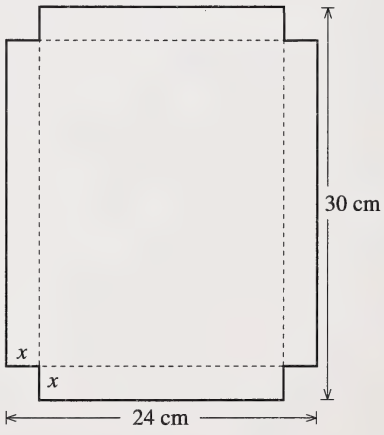
$$12x^2+5x-3=(4x+3)(3x-1)$$

Let $(4x+3)(3x-1)=0$.

$$\begin{array}{lll} 4x+3=0 & \text{or} & 3x-1=0 \\ 4x=-3 & & 3x=1 \\ x=-\frac{3}{4} & & x=\frac{1}{3} \end{array}$$

The roots are -1, $-\frac{3}{4}$, and $\frac{1}{3}$. **(3 marks)**

7. Let x represent the side of the square to be removed.



length $=(30-2x)$ cm
width $=(24-2x)$ cm

The volume of the box expressed as a function is as follows:

$$V=x(30-2x)(24-2x)$$

Since the volume is $1296\ \text{cm}^3$,

$$x(30-2x)(24-2x)=1296$$
$$x\left(720-108x+4x^2\right)=1296$$
$$4x^3-108x^2+720x-1296=0$$
$$x^3-27x^2+180x-324=0$$

Solve using synthetic division.

$$\begin{array}{c|cccc} -3 & 1 & -27 & 180 & -324 \\ & & -3 & 72 & -324 \\ \hline & 1 & -24 & 108 & \boxed{0} \end{array}$$

Since $x-3=0$, then $x=3$.

$Q(x)=x^2-24x+108$

Solve $x^2-24x+108=0$ by factorization.

$(x-6)(x-18)=0$

$x-6=0$ or $x-18=0$

$x=6$ $\quad$ $x=18$ (impossible)

Squares with 3 cm sides or 6 cm sides can be removed. **(5 marks)**

8. If 1 is a zero, then $x-1$ is a factor. If -3 is a zero, then $x-(-3)=x+3$ is a factor. If $\frac{1}{5}$ is a zero, then $\left(x-\frac{1}{5}\right)$ is a factor. Thus, use $5x-1$ for a factor.

$P(x)=k(x-1)(x+3)(5x-1)$

Substitute $P(2)=90$ into the equation.

$P(2)=k(2-1)(2+3)(5(2)-1)$

$90=k(1)(5)(9)$

$90=45k$

$k=2$

$\therefore P(x)=2\left(5x^3+9x^2-17x+3\right)$

$=10x^3+18x^2-34x+6$ **(3 marks)**

9. If $\frac{3}{5}$ is a zero, then $5x-3$ is a factor. If $-\frac{1}{2}$ is a zero, then $2x+1$ is the factor. If -1 is a zero, then $(x+1)(x+1)$ are the factors since -1 has a multiplicity of two.

$P(x)=k(5x-3)(2x+1)(x+1)(x+1)$

$P(0)6=k(0-3)(0+1)(0+1)(0+1)$

$6=k(-3)(1)(1)(1)$

$6=k(-3)$

$k=-2$

$\therefore P(x)=-2\left(10x^4+19x^3+5x^2-7x-3\right)$

$=-20x^4-38x^3-10x^2+14x+6$ **(3 marks)**

Section 4: Graphing Polynomial Functions

Key Concepts

- sketching the graph of a given polynomial function
- determining a polynomial function that defines a given graph
- determining the possible sums of the multiplicities of the zeros of a given real polynomial function
- identifying all zeros of a given real polynomial function

Section 4: Assignment Answer Key (18 marks)

1. a. The y-intercept of the graph is -12. $\left(\frac{1}{2}\text{ mark}\right)$

b. The zeros of $P(x)$ are **1, 3, and -2**. $\left(\frac{1}{2}\text{ mark}\right)$

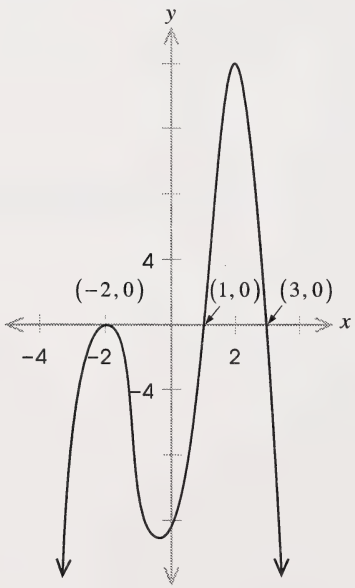
c. The degree of $P(x)$ is **4**. $\left(\frac{1}{2}\text{ mark}\right)$

d. The initial coefficient a_n is **negative** $\left(\frac{1}{2}\text{ mark}\right)$

e. The graph begins by rising to the right in the third quadrant. **(1 mark)**

f. The graph ends by falling to the right in the fourth quadrant. **(1 mark)**

g.

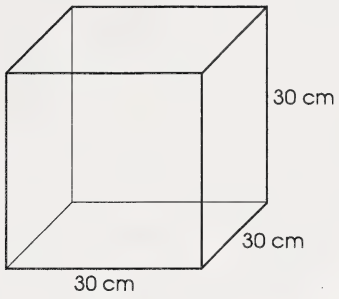


(2 marks)

2. a. $y=(30-2x)(30-2x)(30-2x)$

$=\left(900-120x+4x^2\right)(30-2x)$

$=-8x^3+360x^2-5400x+27\,000$ **(1 mark)**



b. The compartment is jammed with ice. $\left(\frac{1}{2}\text{ mark}\right)$

c. There is a restriction on x.

Domain: $0\le x\le 15$ $\left(\frac{1}{2}\text{ mark}\right)$

d. $-8x^3+360x^2-5400x+27\,000=8000$

$-8x^3+360x^2-5400x+19\,000=0$

$x^3-45x^2+675x-2375=0$

Let $P(x)=x^3-45x^2+675x-2375$. One potential zero of $P(x)$ is 5. Substitute $x=5$ into $P(x)$.

$P(5)=(5)^3-45(5)^2+675(5)-2375$

$=125-1125+3375-2375$

$=0$

Therefore, the thickness of the ice is 5 cm. **(2 marks)**

3. Since the graph touches the x-axis at -2, the zero -2 has a multiplicity of two. The graph also intersects the x-axis at 2 and 3.

$P(x)=k(x+2)^2(x-2)(x-3)$

Since $P(0)=48$.

$48=k(0+2)^2(0-2)(0-3)$

$48=24k$

$k=2$

$P(x)=2(x+2)^2(x-2)(x-3)$

$=2\left(x^2+4x+4\right)\left(x^2-5x+6\right)$

$=8x^4-2x^3-20x^2+8x+48$ **(4 marks)**

4. If -1.5 is a zero, $\left(x+\frac{3}{2}\right)$ is a factor. A factor of $\left(x+\frac{3}{2}\right)$ is equivalent to a factor of $2x+3$ since both give a zero of -1.5.

Since the graph touches at $(2,0)$, the factors are $(x-2)^2$.

$$\therefore P(x)=k(2x+3)(x-2)^2$$
$$P(-1)=k(-2+3)(-1-2)^2$$
$$-27=k(1)(9) \quad \text{(since } P(-1)=-27\text{)}$$
$$-27=9k$$
$$k=-3$$

$$P(x)=-3(2x+3)(x-2)^2$$
$$=-3(2x+3)\left(x^2-4x+4\right)$$
$$=-3\left(2x^3-5x^2-4x+12\right)$$
$$=-6x^3+15x^2+12x-36 \quad \textbf{(4 marks)}$$

Final Module Assignment Answer Key (30 marks)

Part A: Multiple Choice

Each answer in this part is worth 1 mark.

1. B **2.** A **3.** C **4.** D

5. A **6.** C **7.** B **8.** D

(8 marks)

Part B: Numerical Response

Each answer in this part is worth 1 mark.

1. | 1 | 2 | | |

2. | 9 | | | |

(2 marks)

Part C: Written Response

1. $3t^3-8t^2-3t=0$

$t\left(3t^2-8t-3\right)=0$

$t(3t+1)(t-3)=0$

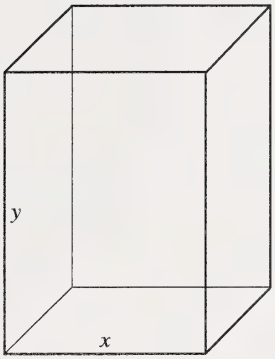
$t=0$ or $3t+1=0$ or $t-3=0$

$t=-\frac{1}{3}$ $t=3$

The object is stopped when $t=0$ s and $t=3$ s. **(2 marks)**

2. a. $x^2 y = 12$ $\qquad A = 4xy + x^2$ (2)

$$y = \frac{12}{x^2} \quad (1)$$



Substitute (1) into (2).

$$A = 4x\left(\frac{12}{x^2}\right) + x^2$$

$$= \frac{48}{x} + x^2 \quad \textbf{(1 mark)}$$

b.

$$A = \frac{48}{x} + x^2$$

$$28 = \frac{48}{x} + x^2$$

$$x^3 - 28x + 48 = 0$$

Solve using the Factor Theorem.

Let $P(x) = x^3 - 28x + 48$. One potential zero of $P(x)$ is 2. Substitute $x = 2$ into $P(x)$.

$$P(2) = (2)^3 - 28(2) + 48$$
$$= 8 - 56 + 48$$
$$= 0$$

Since $P(2) = 0$, $x - 2$ is a factor and $x = 2$.

Use the Factor Theorem again to find the other factors. Another potential zero of $P(x)$ is 4. Substitute $x = 4$ into $P(x)$.

$$P(4) = (4)^3 - 28(4) + 48$$
$$= 64 - 112 + 48$$
$$= 0$$

Since $P(4) = 0$, $x - 4$ is a factor and $x = 2$.

Another potential zero of $P(x)$ is -6. Substitute $x = -6$ into $P(x)$.

$$P(-6) = (-6)^3 - 28(-6) + 48$$
$$= -216 + 168$$
$$= 0$$

Since $P(-6) = 0$, $x + 6$ is a factor and $x = -6$.

The value of -6 is impossible because you cannot have a negative length.

When $x=2$, $y=\frac{12}{x^2}$

$=\frac{12}{(2)^2}$

$=3$

When $x=4$, $y=\frac{12}{(4)^2}$

$=\frac{3}{4}$

Therefore, the dimensions of the box are 2 cm by 2 cm by 4 cm or 4 cm by 4 cm by $\frac{3}{4}$ cm. **(2 marks)**

3. $P(x)=ax^3+bx^2-6$

$P(-2)=a(-2)^3+b(-2)^2-6$

$-38=-8a+4b-6$

$8a-4b=32$

$2a-b=8$ (1)

$P(x)=ax^3+bx^2-6$

$P(3)=a(3)^3+b(3)^2-6$

$12=27a+9b-6$

$27a+9b=18$

$3a+b=2$ (2)

$2a-b=8$ (1)

$3a+b=2$ (2)

(1) + (2): $5a=10$

$a=2$

Substitute $a=2$ in (2).

$3(2)+b=2$

$6+b=2$

$b=-4$

$(a,b)=(2,-4)$ **(4 marks)**

4. $P(x)=(x+1)(x-2)(x-3)$

$=\left(x^2-x-2\right)(x-3)$

$=x^3-4x^2-x+6$

Since the leading coefficient is 2,

$P(x)=2\left(x^3-4x^2-x+6\right)$

$=2x^3-8x^2-2x+12$

Therefore, $a=-8$, $b=-2$, and $c=12$. **(3 marks)**

5. a. The least possible degree is 3.
The leading coefficient is negative.
The zero -1 has a multiplicity of two, and the zero -2 has a multiplicity of one.
(2 marks)

b. The least possible degree is 4.
The leading coefficient is positive.
The zero -2 has a multiplicity of three, and the zero 1 has a multiplicity of one.
(2 marks)

6. The potential zeros of $P(x)$ are $\pm 1,\ \pm\frac{1}{2},\ \pm\frac{1}{4},\ \pm 3,\ \pm\frac{3}{2},\ \pm\frac{3}{4}$.

Oksanna's potential zeros were incorrect. They should be $\pm\frac{3}{2}$.

$$\begin{array}{r|rrrr} 1 & 4 & -4 & -7 & 4 & 3 \\ & & 4 & -8 & 1 & 3 \\ \hline & 4 & -8 & 1 & 3 & \boxed{0} \end{array}$$

Thus, $x=-1$ is a zero and $x+1$ is a factor.

$$\begin{array}{r|rrrr} -\frac{3}{2} & 4 & -8 & 1 & 3 \\ & & -6 & 3 & 3 \\ \hline & 4 & -2 & -2 & \boxed{0} \end{array}$$

Thus, $x=\frac{3}{2}$ is a zero and $2x-3$ is a factor.

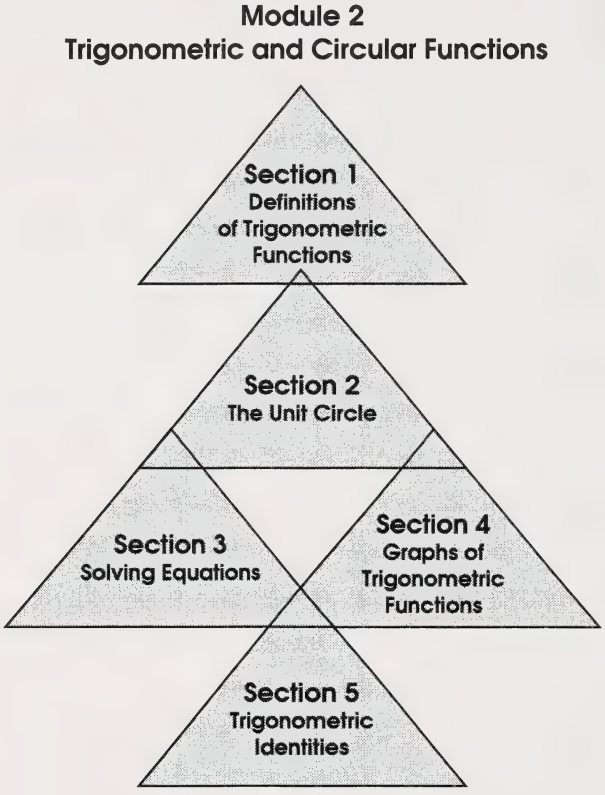
$$\begin{aligned} Q(x) &= \left(4x^2-2x-2\right)\div 2 \\ &= 2x^2-x-1 \\ &= (2x+1)(x-1) \end{aligned}$$

$P(x)=(x+1)(x-1)(2x+1)(2x-3)$ **(4 marks)**

Module 2: Trigonometric and Circular Functions

Overview

Trigonometry is an important and powerful tool for science and engineering. The study of trigonometry in Mathematics 30 emphasizes the understanding of trigonometric ratios and how to use these ratios to solve real-world problems and to study functions. The trigonometric and circular functions, with their relationship with geometry and algebra, are important in the development of matrix representations of rotations, direction angles of vectors, polar coordinates, trigonometric representations of complex numbers, and so on.



Evaluation

The evaluation of this module is based on five section assignments and a final module assignment:

Section 1 Assignment	**13 marks**
Section 2 Assignment	**14 marks**
Section 3 Assignment	**13 marks**
Section 4 Assignment	**14 marks**
Section 5 Assignment	**16 marks**
Final Module Assignment	**30 marks**
TOTAL	**100 marks**

Mandatory Video

Section 4: Activity 1

Graphing Trigonometric Functions I, *Graphing Mathematical Concepts* series, ACCESS Network, available from the Learning Resources Distributing Centre.

Section 4: Activity 3

Graphing Trigonometric Functions II, *Graphing Mathematical Concepts* series, ACCESS Network, available from the Learning Resources Distributing Centre.

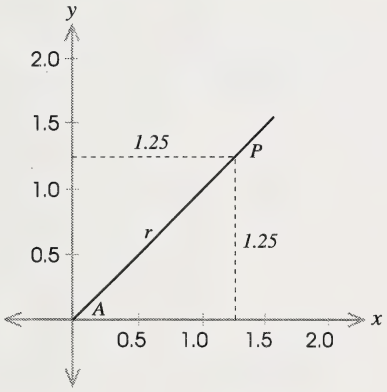
Section 1: Definitions of Trigonometric Functions

Key Concepts

- defining and identifying the values of trigonometric ratios for angles in standard position in terms of x, y, and r
- deriving the values of any trigonometric ratio given the value of one of the others
- determining the reference angle of any given angle and defining the value of a trigonometric ratio for an angle in terms of the reference angle

Section 1: Assignment Answer Key (13 marks)

1. a.



Use the Pythagorean Theorem.

$$r^2 = x^2 + y^2$$
$$r^2 = (1.25)^2 + (1.25)^2$$
$$r^2 = 1.5625 + 1.5625$$
$$r^2 = 3.125$$
$$r = \sqrt{3.125}$$
$$\doteq 1.7678$$

Method 1

$$\sin A \doteq \frac{1.25}{1.7678}$$
$$\doteq 0.7071$$

Method 2

Since the triangle is isosceles (two sides are equal), then $\angle A = 45°$.

$\therefore \sin A \doteq 0.7071$

Therefore, $\sin A = 0.7071$, correct to four decimal places. **(1 mark)**

b. **Method 1**

$$\cos A \doteq \frac{1.25}{1.7678}$$
$$\doteq 0.7071$$

$$\sec A = \frac{1}{\cos A}$$
$$\doteq \frac{1}{0.7071}$$
$$\doteq 1.4142$$

Therefore, $\sec A = 1.4142$, correct to four decimal places. **(1 mark)**

Method 2

$$\sec A = \frac{1}{\cos A}$$
$$\therefore \sec A \doteq \frac{1}{0.7071}$$
$$\doteq 1.4142$$

2. **a.** $\tan A = \frac{12.23}{5.46}$

$\tan A \doteq 2.2399$

$A \doteq 65.942°$

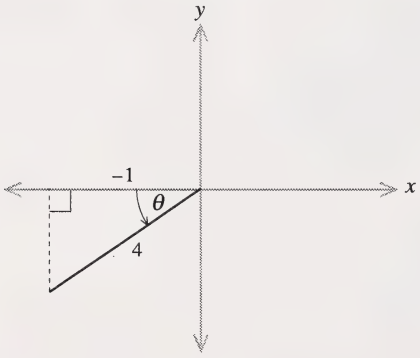
$\sin 65.942° \doteq 0.913$ $\left(1\frac{1}{2}\text{ marks}\right)$

b. $\cos 65.942° \doteq 0.408$ $\left(\frac{1}{2}\text{ mark}\right)$

3. $\tan 302° = \tan(360° - 58°)$

$= -\tan 58°$ (tan is negative in the fourth quadrant.) **(1 mark)**

4. In order for cosine to be negative and tangent to be positive, the angle must be in the third quadrant.

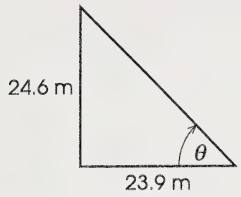


Use the Pythagorean Theorem.

$$r^2 = x^2 + y^2$$
$$4^2 = (-1)^2 + y^2$$
$$y^2 = 16 - 1$$
$$y^2 = 15$$
$$y = -\sqrt{15}$$ (The value is negative since it is in the third quadrant.)

$$\csc \theta = \frac{AC}{AB}$$
$$= \frac{4}{-\sqrt{15}}$$
$$= -\frac{4\sqrt{15}}{15}$$ **(2 marks)**

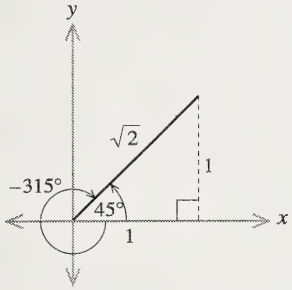
5.



$$\tan\theta = \frac{24.6}{23.9}$$
$$\tan\theta \doteq 1.0293$$
$$\theta \doteq 45.83°$$

The sun's elevation is approximately 45.83°. **(1 mark)**

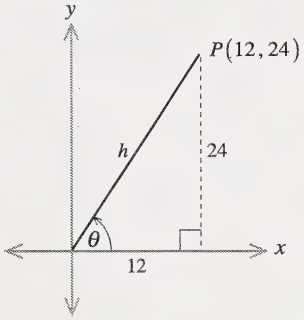
6.



$$\sec(-315°) = \sec 45°$$
$$= \frac{\sqrt{2}}{1}$$
$$= \sqrt{2}$$ **(1 mark)**

7.

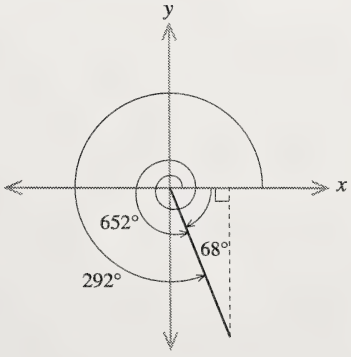
$$h = \sqrt{24^2 + 12^2}$$
$$= \sqrt{720}$$
$$= 12\sqrt{5}$$



$$\sin\theta = \frac{24}{12\sqrt{5}}$$
$$= \frac{2\sqrt{5}}{5}$$

$$\sec\theta = \frac{12\sqrt{5}}{12}$$
$$= \sqrt{5}$$ **(2 marks)**

8. a.

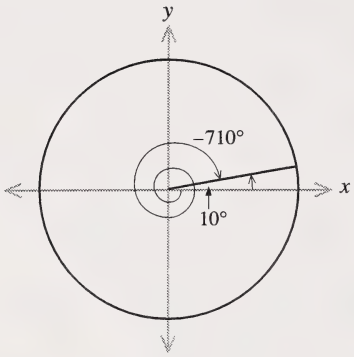


The angle between 0° and 360° coterminal with 652° is $652° - 360° = 292°$.

$$\therefore R = 360° - 292°$$
$$= 68°$$

The angle terminates in the fourth quadrant, and the reference angle is 68°. **(1 mark)**

b.



The angle between 0° and 360° coterminal with $-710°$ is $-710° + 2(360°) = 10°$.

$\therefore\ R = 10°$

The angle terminates in the first quadrant, and the reference angle is 10°. **(1 mark)**

Section 2: The Unit Circle

Key Concepts

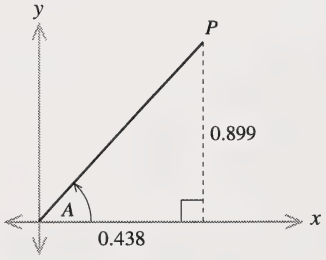
- relating trigonometric functions to coordinates of points on a unit circle
- defining and using radian measure
- examining the relationship between radian and degree measures
- determining and using the exact relative measures of the sides of 30°–60°–90° and 45°–45°–90° triangles
- deriving the exact values of trigonometric ratios for angles with radian measure $\frac{n\pi}{6}$ or $\frac{n\pi}{4}$

Section 2: Assignment Answer Key (14 marks)

1. $\tan A = \dfrac{0.899}{0.438}$ or $\sin A = 0.899$ or $\cos A = 0.438$

$\tan A \doteq 2.0525$ $A \doteq 64°$ $A \doteq 64°$ **(1 mark)**

$A \doteq 64°$



2. The measure of the angle is $\frac{9}{2}\pi$ rad. **(1 mark)**

3. a. $2.6\pi = (2.6\pi)\left(\dfrac{180°}{\pi}\right)$

$= (2.6)(180°)$

$= 468°$ **(1 mark)**

b. $4.2\pi = (4.2\pi)\left(\dfrac{180°}{\pi}\right)$

$= (4.2)(180°)$

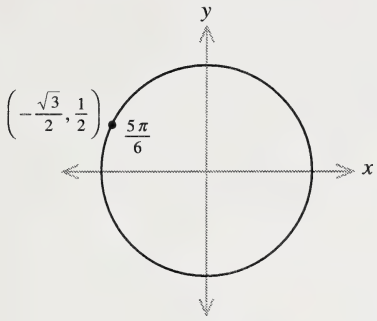
$= 756°$ **(1 mark)**

4. $0.1 \text{ rad} = \frac{180°}{\pi} \times 0.1$

$\doteq 5.7296°$

$\sin 5.7296° \doteq 0.0998$ **(1 mark)**

5.



$$\tan \frac{5\pi}{6} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}}$$

$$= -\frac{1}{\sqrt{3}}$$

$$= -\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= -\frac{\sqrt{3}}{3}$$ **(1 mark)**

6. $\theta = \frac{a}{r}$

$1.9 = \frac{a}{32}$

$a = 60.8$

The distance travelled by the point on the rim of the wheel is 60.8 cm. **(2 marks)**

7. **Method 1**

$$x^2 + y^2 = 1$$

$$\left(-\frac{3}{4}\right)^2 + y^2 = 1$$

$$\frac{9}{16} + y^2 = 1$$

$$y^2 = 1 - \frac{9}{16}$$

$$y^2 = \frac{7}{16}$$

$$y = \frac{\sqrt{7}}{4}$$

$$\tan \theta = \frac{y}{x}$$

$$= \frac{\frac{\sqrt{7}}{4}}{-\frac{3}{4}}$$

$$= -\frac{\sqrt{7}}{3}$$ **(1 mark)**

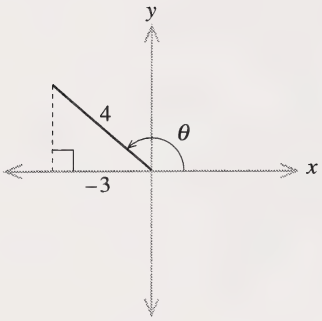
$$\csc \theta = \frac{1}{\sin \theta}$$

$$= \frac{1}{\frac{\sqrt{7}}{4}}$$

$$= \frac{4}{\sqrt{7}}$$

$$= \frac{4\sqrt{7}}{7}$$ **(1 mark)**

Method 2



$$x^2 + y^2 = r^2$$
$$(-3)^2 + y^2 = 4^2$$
$$9 + y^2 = 16$$
$$y^2 = 7$$
$$y = \sqrt{7}$$

$$\tan\theta = \frac{\sqrt{7}}{-3}$$
$$= -\frac{\sqrt{7}}{3} \quad \textbf{(1 mark)}$$

$$\csc\theta = \frac{4}{\sqrt{7}}$$
$$= \frac{4\sqrt{7}}{7} \quad \textbf{(1 mark)}$$

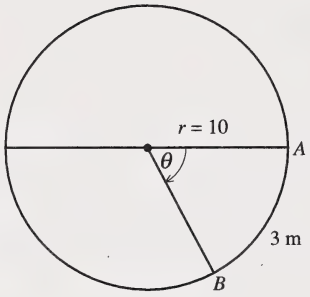
8. $\theta = \frac{a}{r}$

$= \frac{3}{10}$

$= 0.3$ rad

$0.3 \text{ rad} = 0.3 \times \frac{180°}{\pi}$

$\doteq 17.2°$



The central angle is about 0.3 rad or 17.2°. **(2 marks)**

9. $60° = 60° \times \frac{\pi}{180°}$

$= \frac{\pi}{3}$ rad

$\theta = \frac{a}{r}$

$r = \frac{a}{\theta}$

$= \frac{47}{\frac{\pi}{3}}$

$\doteq 45$

The length of the pendulum is approximately 45 cm. **(2 marks)**

Section 3: Solving Equations

Key Concepts

- solving first-degree trigonometric equations
- solving second-degree trigonometric equations
- solving trigonometric equations involving multiples of angles

Section 3: Assignment Answer Key (13 marks)

1. a. In the second and third quadrants, secant is negative.

$$\cos A = \frac{1}{\sec A}$$

$$\cos A = \frac{1}{-2}$$

$$\cos A = -\frac{1}{2}$$

$$A = \frac{2\pi}{3} \text{ and } \frac{4\pi}{3} \quad \textbf{(1 mark)}$$

b. In the first and second quadrants, sine is positive.

$$\therefore A = \frac{\pi}{3} \text{ and } \frac{2\pi}{3} \quad \textbf{(1 mark)}$$

c. In the second and fourth quadrants, tangent is negative.

$$\therefore A = \frac{3\pi}{4} \text{ and } \frac{7\pi}{4} \quad \textbf{(1 mark)}$$

2. a. $8 \sin \theta \tan \theta - 4 \tan \theta = 0$

$4 \tan \theta (2 \sin \theta - 1) = 0$ (Factorize.)

$\therefore 4 \tan \theta = 0$ or $2 \sin \theta = 0$

$\tan \theta = 0$ $\quad 2 \sin \theta = 1$

$\theta = 0°$ and $180°$ $\quad \sin \theta = \frac{1}{2}$

$\theta = 30°$ and $150°$

Therefore, $\theta = 0°$, $30°$, $150°$, and $180°$. **(2 marks)**

b. Four angles satisfy the equation. $\left(\frac{1}{2}\textbf{ mark}\right)$

c. The largest quadrantal angle is $180°$. $\left(\frac{1}{2}\textbf{ mark}\right)$

3. $d = \frac{1}{10} {v_0}^2 \sin 2\theta$

$80 = \frac{1}{10}(40)^2 \sin 2\theta$

$80 = 160 \sin 2\theta$

$\sin 2\theta = \frac{1}{2}$

$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, 2\pi + \frac{\pi}{6}$, and $2\pi + \frac{5\pi}{6}$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}$, and $\frac{17\pi}{12}$ **(2 marks)**

4. a.

$$16\tan^2 y = 1 - 33\tan^2 y$$
$$16\tan^2 y + 33\tan^2 y = 1$$
$$49\tan^2 y = 1$$
$$\tan^2 y = \frac{1}{49}$$
$$\tan y = \pm\frac{1}{7}$$
$$y \doteq 8°,\ 188°,\ 172°,\ \text{and } 352° \quad \left(1\tfrac{1}{2}\ \textbf{marks}\right)$$

b. Four angles satisfy the equation. $\left(\frac{1}{2}\ \textbf{mark}\right)$

c. The third-quadrant angle that satisfies the equation is 188°. $\left(\frac{1}{2}\ \textbf{mark}\right)$

d. The fourth-quadrant angle that satisfies the equation is 352°. $\left(\frac{1}{2}\ \textbf{mark}\right)$

5.

$$2\cos^2\theta - 3\cos\theta + 1 = 0$$
$$(2\cos\theta - 1)(\cos\theta - 1) = 0 \quad \text{(Factorize using the trinomial method.)}$$

$\therefore\ 2\cos\theta - 1 = 0$ or $\cos\theta - 1 = 0$

$2\cos\theta = 1$ $\cos\theta = 1$

$\cos\theta = \frac{1}{2}$ $\theta = 0°$ and $360°$

$\theta = 60°$ and $300°$

Therefore, $\theta = 0°,\ 60°,\ 300°,$ and $360°$. **(2 marks)**

Section 4: Graphs of Trigonometric Functions

Key Concepts

- graphing the six trigonometric functions and identifying their domain, range, and period
- identifying amplitude, period, and phase shift

Section 4: Assignment Answer Key (14 marks)

1. a. When a person inhales, the maximum volume of air in the lungs is 2.5 L. $\left(\frac{1}{2}\ \textbf{mark}\right)$

b. It takes 6 s (Period: $10 - 4\ \text{s} = 6\ \text{s}$) for a person to inhale and exhale once. $\left(\frac{1}{2}\ \textbf{mark}\right)$

c. A volume of 2.0 L of air (Minimum Value: 2.0 L) constantly remains in the lungs. $\left(\frac{1}{2}\ \textbf{mark}\right)$

d. A volume of 0.5 L of air (Range: $2.5 - 2.0\ \text{L} = 0.5\ \text{L}$) is exhaled. $\left(\frac{1}{2}\ \textbf{mark}\right)$

2. The domain is $\left\{\theta \middle| \theta \neq \frac{\pi}{2} + n\pi,\ n \in I\right\}$.

The period is π.
(1 mark)

3. The domain is $\{\theta | \theta \in R\}$.

The range is $\{y | -1 \leq y \leq 1\}$.
(1 mark)

4.

Parameter	Description	Effects on the Graph of $y = \cos 2\theta$
3	amplitude	The distance between the maximum and minimum values changes from 2 to 6; or, the distance from the midline to the maximum or minimum increases from 1 to 3.
$\frac{\pi}{6}$	horizontal translation	The graph of $y = \cos 2\theta$ moves $\frac{\pi}{6}$ units to the right.
1	vertical translation	The midline of $y = 3\cos\left(2\theta - \frac{\pi}{3}\right)$ moves up one unit

(3 marks)

5. The amplitude is half the difference of the maximum and minimum values.

$$a = \frac{5-(-3)}{2}$$
$$= \frac{8}{2}$$
$$= 4$$

Therefore, the amplitude is 4.

The range is all real numbers from the minimum value to the maximum value. Therefore, the range is $-3 \leq y \leq 5$.

The vertical translation is half the sum of the maximum and minimum values.

$$d = \frac{5-3}{2}$$
$$= 1$$

Therefore, the vertical translation is 1.
(3 marks)

6. a. $a = \frac{M-m}{2}$ $\quad p = \frac{2\pi}{b}$ $\quad d = \frac{M+m}{2}$

$= \frac{120+60}{2}$ $\quad b = \frac{2\pi}{4}$ $\quad = \frac{120-60}{2}$

$= 90$ $\quad = \frac{\pi}{2}$ $\quad = 30$

Therefore, the equation of the graph is $y = 90 \sin \frac{\pi}{2}t + 30$. **(2 marks)**

b. $y = 90 \sin \frac{\pi}{2}(4.25) + 30$

$= 90 \sin 382.5 + 30$

$\doteq 64.4$

The acrobat's feet are approximately 64.4 cm above the level of the trampoline. **(1 mark)**

7. The value of $\frac{5\pi}{4}$ does not change.

The amplitude changes from 4 to 2. It decreases by a factor of $\frac{1}{2}$.

$\therefore \frac{-2\sqrt{2}}{2} = -\sqrt{2}$

The new point of intersection is $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$. **(1 mark)**

Section 5: Trigonometric Identities

Key Concepts

- deriving and applying the quotient identities, the reciprocal identities, and the Pythagorean identities to expand or simplify trigonometric expressions
- deriving and applying the negative-angle relations and complementary-angle relations
- deriving and applying the sum and difference formulas

Section 5: Assignment Answer Key (16 marks)

1. $\sec y = \frac{1}{\cos y}$ $\quad \therefore \frac{\cos y}{\sec y} = \frac{\cos y}{\frac{1}{\cos y}}$

$= \cos y \cos y$

$= \cos^2 y$ **(1 mark)**

2. $\dfrac{\cos y - \sin^2 y}{\sin y} = \dfrac{\cos y}{\sin y} - \dfrac{\sin^2 y}{\sin y}$ (Divide each term in the numerator by $\sin y$.)

$= \cot y - \sin y$ **(2 marks)**

3. $\cos\theta + \tan\theta \sin\theta = \cos\theta + \left(\dfrac{\sin\theta}{\cos\theta}\right)(\sin\theta)$

$= \cos\theta + \dfrac{\sin^2\theta}{\cos\theta}$

$= \dfrac{\cos^2\theta + \sin^2\theta}{\cos\theta}$

$= \dfrac{1}{\cos\theta}$

$= \sec\theta$ **(3 marks)**

4. $\left(1 + \cot^2\theta\right)\cos^2\theta = \csc^2\theta \cos^2\theta$

$= \dfrac{1}{\sin^2\theta}\cos^2\theta$

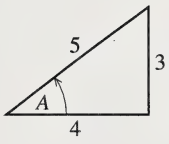
$= \dfrac{\cos^2\theta}{\sin^2\theta}$

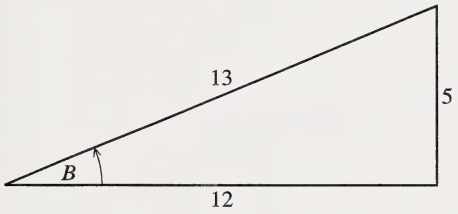
$= \cot^2\theta$ **(3 marks)**

5. $\cot\left(-\frac{\pi}{2} + 2k\right)$ is in the fourth quadrant. Therefore, k must be positive.

$\therefore \cot\left(-\frac{\pi}{2} + 2k\right) = -\tan 2k$ **(1 mark)**

6.





$\sin A = \dfrac{3}{5}$, $\cos A = \dfrac{4}{5}$ $\sin B = \dfrac{5}{13}$, $\cos B = \dfrac{12}{13}$

$\therefore \sin(A+B) = \sin A \cos B + \cos A \sin B$

$= \left(\dfrac{3}{5}\right)\left(\dfrac{12}{13}\right) + \left(\dfrac{4}{5}\right)\left(\dfrac{5}{13}\right)$

$= \dfrac{36}{65} + \dfrac{20}{65}$

$= \dfrac{56}{65}$ **(3 marks)**

7. Method 1

$$\tan 15° = \tan (60° - 45°)$$

$$= \frac{\sin (60° - 45°)}{\cos (60° - 45°)}$$

$$= \frac{\sin 60° \cos 45° - \cos 60° \sin 45°}{\cos 60° \cos 45° + \sin 60° \sin 45°}$$

$$= \frac{\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)}{\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)}$$

$$= \frac{\frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}}{\frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}}$$

$$= \frac{\frac{\sqrt{3}-1}{2\sqrt{2}}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}$$

$$= \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$= \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}$$

$$= \frac{4 - 2\sqrt{3}}{2}$$

$$= 2 - \sqrt{3}$$ **(3 marks)**

Method 2

$$\tan 15° = \tan (60° - 45°)$$

$$= \frac{\tan 60° - \tan 45°}{1 + \tan 60° \tan 45°}$$

$$= \frac{\sqrt{3}-1}{1+\sqrt{3}}$$

$$= \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}$$

$$= \frac{4 - 2\sqrt{3}}{2}$$

$$= 2 - \sqrt{3}$$

Method 3

$$\tan 15° = \tan (45° - 30°)$$

$$= \frac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°}$$

$$= \frac{1 - \frac{\sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}}$$

$$= \frac{3-\sqrt{3}}{3+\sqrt{3}} \times \frac{3-\sqrt{3}}{3-\sqrt{3}}$$

$$= \frac{12 - 6\sqrt{3}}{6}$$

$$= 2 - \sqrt{3}$$

Final Module Assignment Answer Key (30 marks)

Part A: Multiple Choice

Each answer in this part is worth 1 mark.

1. B **2.** A **3.** D **4.** C **5.** D **6.** B
(6 marks)

Part B: Numerical Response

Each answer in this part is worth 1 mark.

1. 5 **2.** 93.6 **3.** 0.34 **4.** 421
(4 marks)

Part C: Written Response

1. $a = \frac{7+3}{2}$
$= 5$

$p = \frac{2\pi}{b}$
$\pi = \frac{2\pi}{b}$
$b = 2$

$d = \frac{7-3}{2}$
$= 2$

$\therefore y = 5\cos 2\theta + 2$

Substitute $(380°, y)$ in $y = 5\cos 2\theta + 2$.

$y = 5\cos 2(380°) + 2$
$= 5.830$ **(4 marks)**

2. **a.** The roots are π and 2π. **(1 mark)**

b. $\sin 2\theta = \sin\theta \cos\theta + \cos\theta \sin\theta$
$= 2\sin\theta \cos\theta$

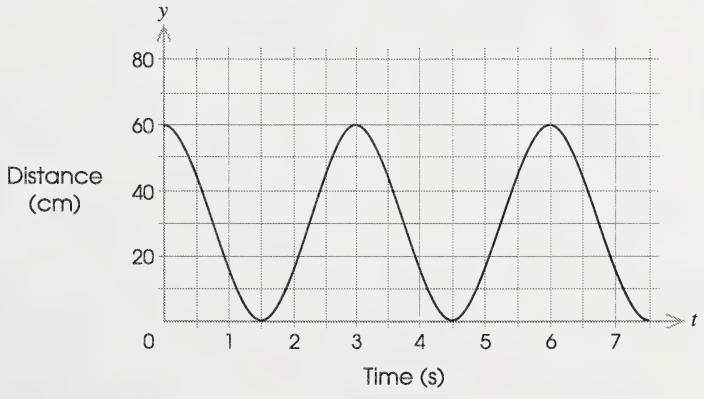
$\sin\theta - \sin 2\theta = 0$
$\sin\theta - 2\sin\cos\theta = 0$
$\sin\theta(1 - 2\cos\theta) = 0$

$\sin\theta = 0$
$\theta = 0, \pi$, and 2π

or $1 - 2\cos\theta = 0$
$\cos\theta = \frac{1}{2}$
$\theta = \frac{\pi}{3}$ and $\frac{5\pi}{3}$

The roots at points P and R are $\frac{\pi}{3}$ and $\frac{5\pi}{3}$ respectively. **(2 marks)**

3. **a.** The maximum and minimum values are 60 cm and 0 cm respectively. P moves to a first minimum position in 1.5 s. Since the nail moves from a maximum point to a minimum point in 1.5 s, then the period is 3 s. **(2 marks)**



b. $a = \frac{M-m}{2}$ $\quad p = \frac{2\pi}{b}$ $\quad d = \frac{M+m}{2}$ $\quad y = 30\cos\frac{2\pi}{3}t + 30$

$= \frac{60-0}{2}$ $\quad 3 = \frac{2\pi}{b}$ $\quad = \frac{60+0}{2}$

$= 30$ $\quad b = \frac{2\pi}{3}$ $\quad = 30$

$= \frac{2\pi}{3}$

(2 marks)

c. $y = 30\cos\frac{2\pi}{3}t + 30$

$= 30\cos\frac{2\pi}{3}(8) + 30$

$= 15$

The nail is 15 cm above the ground. **(1 mark)**

4. $\sqrt{\frac{1+\cot^2\theta}{\csc^4\theta}} = \sqrt{\frac{\csc^2\theta}{\csc^4\theta}}$

$= \sqrt{\frac{1}{\csc^2\theta}}$

$= \frac{1}{\csc\theta}$

$= \sin\theta$ **(2 marks)**

5. a. Compare the graph of $f(\theta) = \cos\theta$ with the graph on the computer display. The graph of $a\cos\theta$ is shifted $\frac{3\pi}{2}$ units to the right. Therefore, the maximum negative value of c is $\frac{-3\pi}{2}$. **(2 marks)**

b. The parameters that affect the range of the cosine function are a and d. The parameter a causes a magnification of the graph, and the parameter d is a vertical translation. **(2 marks)**

6. $\frac{1-\cos^2\theta}{\sec^2\theta - 1} = \frac{\sin^2\theta}{\tan^2\theta}$

$= \frac{\sin^2\theta}{\frac{\sin^2\theta}{\cos^2\theta}}$

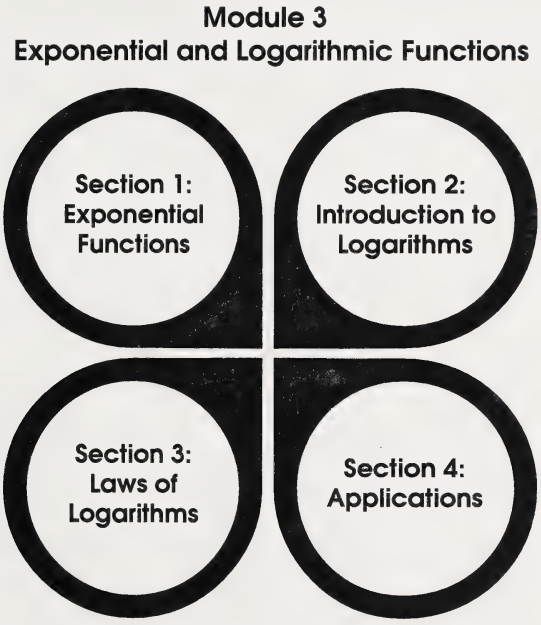
$= \frac{\cancel{\sin^2\theta}}{1} \times \frac{\cos^2\theta}{\cancel{\sin^2\theta}}$

$= \cos^2\theta$ **(2 marks)**

Module 3: Exponential and Logarithmic Functions

Overview

The study of exponential and logarithmic functions is an application of the general study of functions. In this module, students have an opportunity to understand the connections of functions to real-world phenomena. The study of the operations in logarithms and logarithmic equations further provides the student with the opportunity to understand how a mathematics system operates and realize that the rules that govern properties and solutions of equations apply to the system as a whole.



Evaluation

The evaluation of this module is based on four assignments and a final module assignment:

Section 1 Assignment	**7 marks**
Section 2 Assignment	**18 marks**
Section 3 Assignment	**21 marks**
Section 4 Assignment	**24 marks**
Final Module Assignment	**30 marks**
TOTAL	**100 marks**

Optional Video

Section 4: Introduction

Graphing Logarithms and Exponents, *Graphing Mathematical Concepts* series, ACCESS Network, available from Learning Resources Distributing Centre.

Section 1: Exponential Functions

Key Concepts

- maintaining previous skills with exponential expressions
- identifying and graphing exponential functions
- solving exponential equations by transforming both sides to a common base
- solving exponential functions describing problems involving exponential growth and decay

Section 1: Assignment Answer Key (7 marks)

1. a. $8^{\frac{2}{3}} \cdot 16^{-\frac{3}{4}} = \left(2^3\right)^{\frac{2}{3}} \cdot \left(2^4\right)^{-\frac{3}{4}}$

$= 2^{\frac{6}{3}} \cdot 2^{-\frac{12}{4}}$

$= 2^2 \cdot 2^{-3}$

$= 2^{-1}$ or $\frac{1}{2}$ $\left(\frac{1}{2}\ \textbf{mark}\right)$

b. $\left(\frac{a^2 b^{-3}}{b^{-\frac{1}{2}}}\right)^4 = \frac{a^8 b^{-12}}{b^{-2}}$

$= a^8 b^{-10}$ $\left(\frac{1}{2}\ \textbf{mark}\right)$

2. Answers may vary depending on the student's interpretation of the graph. Give full marks if the answers are close to the ones given.

a. $y \doteq 42$ $\left(\frac{1}{2}\ \textbf{mark}\right)$

b. $x \doteq 1.7$ $\left(\frac{1}{2}\ \textbf{mark}\right)$

3. $9^{2x} = \left(3^x\right)^{x-1}$

$\left(3^2\right)^{2x} = \left(3^x\right)^{x-1}$

$3^{4x} = 3^{x^2 - x}$

$4x = x^2 - x$ (Since the bases are equal, equate the exponents.)

$0 = x^2 - 5x$

$0 = x(x-5)$

$x = 0$ or $x - 5 = 0$

$x = 5$ **(1 mark)**

4. $A_0 = 1$, $A = 32$, and $t = 40$

$A = A_0\left(2^{\frac{t}{p}}\right)$

$32 = 1\left(2^{\frac{40}{p}}\right)$

$2^5 = 2^{\frac{40}{p}}$

$5 = \frac{40}{p}$ (Since the bases are equal, equate the exponents.)

$p = 8$

The time required for the bacteria to double is 8 min. **(2 marks)**

5. $N = \frac{1}{\sqrt{2}} N_0$ and $t = 8$

$$N = N_0 \left(\frac{1}{2}\right)^{\frac{t}{h}}$$
$$\frac{1}{\sqrt{2}} N_0 = N_0 \left(\frac{1}{2}\right)^{\frac{8}{h}}$$
$$\frac{1}{\sqrt{2}} = \frac{1}{2^{\frac{8}{h}}}$$
$$2^{-\frac{1}{2}} = 2^{-\frac{8}{h}}$$
$$-\frac{1}{2} = -\frac{8}{h}$$
$$h = 16$$

The half-life is 16 d. **(2 marks)**

Section 2: Introduction to Logarithms

Key Concepts

- defining logarithmic functions
- identifying and graphing logarithmic functions
- transforming an equation in exponential form to logarithmic form and vice versa
- solving equations of the form $y = \log_a (x)$, where $a > 0$ and $a \neq 1$, using exponential form

Section 2: Assignment Answer Key (18 marks)

1. a. $x = b^y$ $\left(\frac{1}{2}\right.$ **mark**$\left.\right)$ **b.** $x = b^y \Leftrightarrow y = \log_b x$ $\left(\frac{1}{2}\right.$ **mark**$\left.\right)$

2. Answers may vary depending on the students interpretation of the graph. Give full marks if the answers are close to the ones given.

a. $\log_3 6 \doteq 1.6$ **(1 mark)** **b.** $\log_3 20 \doteq 2.7$ **(1 mark)** **c.** $x \doteq 16$ **(1 mark)**

3. a. $64 = 4^3$
$\log_4 64 = 3$ **(1 mark)**

b. $2^{-3} = \frac{1}{8}$
$\log_2 \frac{1}{8} = -3$ **(1 mark)**

4. a. $\log_a b = c$
$a^c = b$ $\left(\frac{1}{2}\right.$ **mark**$\left.\right)$

b. $\log_2 \frac{1}{32} = -5$
$\frac{1}{32} = 2^{-5}$ $\left(\frac{1}{2}\right.$ **mark**$\left.\right)$

5. a. Let $x = \log_3 \frac{1}{27}$.

$3^x = \frac{1}{27}$

$3^x = \frac{1}{3^3}$

$3^x = 3^{-3}$

$\therefore x = -3$ **(1 mark)**

b. $\log_3 \sqrt[4]{81^3} = \log_3 \left[\left(3^4\right)^2 \right]^{\frac{1}{4}}$

$= \log_3 3^3$

$= 3$ **(1 mark)**

6. a. $\log_5 5^7 = 7 \log_5 5$

$= 7(1)$

$= 7$ $\left(\frac{1}{2}\right.$ **mark**$\left.\right)$

b. Since $a^{\log_a m} = m$, then $6^{\log_6 14} = 14$. $\left(\frac{1}{2}\right.$ **mark**$\left.\right)$

7. a. $\log_x 64 = -3$

$x^{-3} = 64$

$\frac{1}{x^3} = \frac{64}{1}$

$x^3 = \frac{1}{64}$

$x = \frac{1}{4}$ **(2 marks)**

b. $\log_{32} x = -\frac{3}{5}$

$(32)^{-\frac{3}{5}} = x$

$\frac{1}{(32)^{\frac{3}{5}}} = x$

$\frac{1}{2^3} = x$

$\frac{1}{8} = x$ **(2 marks)**

8. a. Let $x = \log_3 \left(\log_4 64\right)$.

$x = \log_3 \left(\log_4 4^3\right)$

$= \log_3 3$

$= 1$ **(1 mark)**

b. Let $x = \log_4 \left(\log_3 3^2\right)$.

$x = \log_4 2$ (since $\log_3 3^2 = 2$)

$4^x = 2$

$\left(2^2\right)^x = 2^1$

$2x = 1$

$x = \frac{1}{2}$

Thus, $\log_4 \left(\log_3 3^2\right) = \frac{1}{2}$. **(2 marks)**

9. a. $\log 18 = 1.255\ 27$ $\left(\frac{1}{2}\right.$ **mark**$\left.\right)$

b. $10^{2.46} = 288.403\ 15$ $\left(\frac{1}{2}\right.$ **mark**$\left.\right)$

Section 3: Laws of Logarithms

Key Concepts

- deriving the laws of logarithms from the exponent laws
- applying the laws of logarithms in simplifying or evaluating expressions involving logarithms

Section 3: Assignment Answer Key (21 marks)

1. a. $\log_3 \frac{\sqrt[4]{AB}}{C} = \frac{1}{4}\log_3 A + \frac{1}{4}\log_3 B - \log_3 C$ **(2 marks)**

b. $\log_4 \frac{AB^{\frac{2}{3}}}{C^{\frac{1}{3}}} = \log_4 A + \frac{2}{3}\log_4 B - \frac{1}{3}\log_4 C$ **(2 marks)**

2. a. $\frac{1}{4}\log_7 A + \frac{1}{2}\log_7 B - \frac{1}{6}\log_7 C = \log_7 \frac{A^{\frac{1}{4}}B^{\frac{1}{2}}}{C^{\frac{1}{6}}}$ or $\log_7 \frac{\sqrt[4]{A}\sqrt{B}}{\sqrt[6]{C}}$ **(2 marks)**

b. $\frac{1}{3}\left(\log_5 A + 2\log_5 B\right) - 3\left(\log_5 C + \log_5 D\right) = \log_5 \frac{\left(AB^2\right)^{\frac{1}{3}}}{(CD)^3}$ or $\log_5 \frac{\sqrt[3]{AB^2}}{C^3 D^3}$ **(2 marks)**

3. a.
$$\begin{aligned}\log_4 8 + 2\log_4 3 - \log_4 12 &= \log_4 \frac{8(3)^2}{12}\\ &= \log_4 \frac{8(9)}{12}\\ &= \log_4 6 \quad \textbf{(2 marks)}\end{aligned}$$

b.
$$\begin{aligned}2\log_3 7 + 3\log_3 2 - 2\log_3 2 &= \log_3 \frac{7^2(2)^3}{2^2}\\ &= \log_3 \frac{49(8)}{4}\\ &= \log_3 98 \quad \textbf{(2 marks)}\end{aligned}$$

4.
$$\begin{aligned}\log_3 \frac{5589}{69} &= \log_3 81\\ &= \log_3 3^4\\ &= 4 \quad \textbf{(1 mark)}\end{aligned}$$

5.
$$\begin{aligned}\log 60.32 - \log 9.84 - \log 5.68 &= \log \frac{(60.32)}{(9.84)(5.68)}\\ &\doteq \log 1.079\,239\\ &\doteq 0.033 \quad \textbf{(1 mark)}\end{aligned}$$

6. a. Let $x = \log_2 10 + \log_2 3.2$.

$$x = \log_2 (10 \times 3.2)$$
$$= \log_2 32$$
$$= 5 \quad \textbf{(1 mark)}$$

b. Method 1

$$\log_8 2\left(\sqrt{32}\right) = \log_8 2 + \log_8 32^{\frac{1}{2}}$$

Let $x = \log_8 2$ and $y = \log_8 32^{\frac{1}{2}}$.

$x = \log_8 2$	$y = \log_8 32^{\frac{1}{2}}$
$8^x = 2$	$8^y = 32^{\frac{1}{2}}$
$2^{3x} = 2$	$2^{3y} = 2^{\frac{5}{2}}$
$3x = 1$	$3y = \frac{5}{2}$
$x = \frac{1}{3}$	$y = \frac{5}{6}$

$$x + y = \frac{1}{3} + \frac{5}{6}$$
$$= \frac{7}{6}$$

Thus, $\log_8 2\left(\sqrt{32}\right) = \frac{7}{6}$. **(2 marks)**

Method 2: Change of Base Formula

$\log_a b = \dfrac{\log_c b}{\log_c a}$, where c equals 10.

$$\log_8 (2)\left(\sqrt{32}\right) = \frac{\log_{10} (2)\left(\sqrt{32}\right)}{\log_{10} 8}$$
$$= \frac{\log_{10}\left[2\left(2^{\frac{5}{2}}\right)\right]}{\log_{10} 2^3}$$
$$= \frac{\log_{10} 2^{\frac{7}{2}}}{\log_{10} 2^3}$$
$$= \frac{\frac{7}{2}\log_{10} 2}{3\log_{10} 2}$$
$$= \frac{\frac{7}{2}}{3}$$
$$= \frac{7}{6}$$

7. $\log_5 (15)^{\frac{2}{3}} = \frac{2}{3}\log_5 15$

$$= \frac{2}{3}\log_5 (3 \times 5)$$
$$= \frac{2}{3}\left(\log_5 3 + \log_5 5\right)$$
$$\doteq \frac{2}{3}(0.682\,606 + 1)$$
$$\doteq \frac{2}{3}(1.682\,606)$$
$$\doteq 1.121\,74 \quad \textbf{(2 marks)}$$

8. a. $\frac{1}{5}\log 23.8 \doteq \frac{1}{5}(1.376\,5769)$

$$\doteq 0.2753$$
$$\doteq 0.275 \quad \textbf{(1 mark)}$$

b. $\frac{3}{5}\log 8.7 + \frac{1}{4}\log 4.2 \doteq 0.5637 + 0.1558$

$$\doteq 0.7195$$
$$\doteq 0.720 \quad \textbf{(1 mark)}$$

Section 4: Applications

Key Concepts

- solving exponential equations
- solving logarithmic equations
- applying exponential and logarithmic functions to solve problems

Section 4: Assignment Answer Key (24 marks)

1. a. $\frac{1}{3}x\log 5 = \log 18$

$$x = \frac{3\log 18}{\log 5}$$
$$\doteq \frac{3.765\,818}{0.698\,970}$$
$$\doteq 5.387\,67 \quad \textbf{(2 marks)}$$

b.
$$4x\log 8 = (2x+2)\log 5$$
$$4x\log 8 = 2x\log 5 + 2\log 5$$
$$4x\log 8 - 2x\log 5 = 2\log 5$$
$$x(4\log 8 - 2\log 5) = 2\log 5$$
$$x = \frac{2\log 5}{4\log 8 - 2\log 5}$$
$$\doteq \frac{1.397\,940}{3.612\,360 - 1.397\,940}$$
$$\doteq 0.631\,29 \quad \textbf{(2 marks)}$$

2. The change of base formula is $\log_a b = \frac{\log_c b}{\log_c a}$, where c equals 10.

$$\log_6 72 = \frac{\log 72}{\log 6}$$
$$\doteq \frac{1.857\,332}{0.778\,151}$$
$$\doteq 2.386\,85 \quad \textbf{(2 marks)}$$

3. $\log x^{\frac{2}{3}} = \log 24$

$$\frac{2}{3}\log x = \log 24$$
$$\log x = \frac{3}{2}\log 24$$
$$\log x \doteq \frac{3}{2}(1.380\,211\,242)$$
$$\log x \doteq 2.070\,316\,863$$
$$x \doteq 117.576 \quad \textbf{(2 marks)}$$

4.
$$\log\frac{x}{4} = \log[(7x)(2x)]$$
$$\frac{x}{4} = 14x^2$$
$$56x^2 = x$$
$$56x^2 - x = 0$$
$$x(56x-1) = 0$$

$x = 0$ or $56x - 1 = 0$

$$x = \frac{1}{56}$$

The value $x = 0$ is inadmissible. Thus, $x = \frac{1}{56}$. **(2 marks)**

5.
$$\log_4[(x-1)(x+2)] = 1$$
$$\log_4(x-1)(x+2) = \log_4 4$$
$$(x-1)(x+2) = 4 \quad \text{(If } \log_x a = \log_x b, \text{ then } a = b.\text{)}$$
$$x^2 + x - 2 = 4$$
$$x^2 + x - 6 = 0$$
$$(x-2)(x+3) = 0$$

$x - 2 = 0$ or $x + 3 = 0$

$x = 2$ $\qquad$ $x = -3$

The value $x = -3$ is inadmissible. Thus, $x = 2$. **(2 marks)**

6. A formula for predicting population growth is $G = G_0\left(2^{cy}\right)$ where G is the population at time y in years, G_0 is the population when $y = 0$, and c is a constant that is always positive.

$$G = G_0\left(2^{cy}\right)$$
$$47\,700 = 41\,300\left(2^{c(8)}\right)$$
$$\frac{47\,700}{41\,300} = 2^{8c}$$
$$\log\frac{47\,700}{41\,300} = 8c\log 2$$
$$c = \frac{\log\frac{47\,700}{41\,300}}{8\log 2}$$
$$\doteq \frac{0.062\,568}{2.408\,240}$$
$$\doteq 0.0260$$

For the year 1995, $G_0 = 41\,300$, $c = 0.0260$, and $y = 19$.

$$G \doteq 41\,300\left(2^{0.0260}\right)^{(19)}$$
$$\doteq 58\,164$$

The population in 1995 will be about 58 000. **(3 marks)**

7. The half-life is the time required for the substance to decay to one-half of its original amount. The formula $a = a_0\left(2^{-\frac{t}{h}}\right)$ represents the amount a remaining after time t. The original amount is a_0, and h is the half-life of the substance.

Use the formula $a = a_0\left(2^{-\frac{t}{h}}\right)$.

$a = 30$, $a_0 = 200$, and $t = 40$

$$2^{-\frac{t}{h}} = \frac{1}{2^{\frac{t}{h}}} = \left(\frac{1}{2}\right)^{\frac{t}{h}}$$

$$a = a_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

$$30 = 200\left(\frac{1}{2}\right)^{\frac{40}{h}}$$

$$\frac{30}{200} = \left(\frac{1}{2}\right)^{\frac{40}{h}}$$

$$\log\frac{3}{20} = \frac{40}{h}\log\frac{1}{2}$$

$$h\log\frac{3}{20} = 40\log\frac{1}{2}$$

$$h = \frac{40\log\frac{1}{2}}{\log\frac{3}{20}}$$

$$\doteq \frac{-12.041\,199}{-0.823\,908}$$

$$\doteq 14.6$$

The half-life, to one decimal place, is 14.6 d. **(3 marks)**

8. $P = 1200$, $r = 0.009$, $n = 2$, and $t = 6$

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$= 1200\left(1+\frac{0.09}{2}\right)^{2(6)}$$

$$= 1200(1+0.045)^{12}$$

$$= 1200(1.045)^{12}$$

$$\doteq \$2035.06$$ **(3 marks)**

9. The Richter scale uses the formula $\log_{10} \frac{I_m}{I_r}$, where I_m is the intensity of the earthquake being measured and I_r is the intensity of a reference earthquake.

Earthquake A

$$5.8 = \log_{10} \frac{I_m}{I_r}$$

$$10^{5.8} = \frac{I_m}{I_r}$$

$$I_m = 10^{5.8} I_r$$

Earthquake B

$$5.1 = \log_{10} \frac{I_m}{I_r}$$

$$10^{5.1} = \frac{I_m}{I_r}$$

$$I_m = 10^{5.1} I_r$$

$$\frac{I_m \text{ for Earthquake A}}{I_m \text{ for Earthquake B}} = \frac{10^{5.8} I_r}{10^{5.1} I_r}$$

$$= 10^{5.8-5.1}$$

$$= 10^{0.7}$$

$$\doteq 5$$

Earthquake A is about 5 times more intense than Earthquake B. **(3 marks)**

Final Module Assignment (30 marks)

Part A: Multiple Choice

Each answer in this part is worth 1 mark.

1. A **2.** C **3.** B **4.** D **5.** C **6.** A **7.** B

(7 marks)

Part B: Numerical Response

Each answer in this part is worth 1 mark.

1. 3.67 **2.** 3 **3.** 1.7

(3 marks)

Part C: Written Response

1.

$$\log_3 81 = 2x^2 + 7x - 11$$
$$\log_3 3^4 = 2x^2 + 7x - 11$$
$$4 = 2x^2 + 7x - 11$$
$$2x^2 + 7x - 15 = 0$$
$$(2x-3)(x+5) = 0$$

$2x-3=0$ or $x+5=0$

$x = \frac{3}{2}$ $\quad$ $x = -5$

The values of x are $\frac{3}{2}$ and -5. **(2 marks)**

2.

$$a = a_0\left(2^{-\frac{t}{h}}\right)$$
$$0.25 = 4.0\left(\frac{1}{2}\right)^{\frac{t}{3.8}}$$
$$\frac{0.25}{4} = \left(\frac{1}{2}\right)^{\frac{t}{3.8}}$$
$$\log\frac{0.25}{4} = \frac{t}{3.8}\log\frac{1}{2}$$
$$\frac{t}{3.8} = \frac{\log\frac{0.25}{4}}{\log\frac{1}{2}}$$
$$t \doteq \frac{-1.204\,12}{-0.301\,030}$$
$$\doteq 15.2$$

It will take radon about 15.2 d to decay to 0.25 g. **(4 marks)**

3.

$$\log_2(x-3) + \log_2(x-4) = \log_2 2$$
$$(x-3)(x-4) = 2$$
$$x^2 - 7x + 12 = 2$$
$$x^2 - 7x + 10 = 0$$
$$(x-5)(x-2) = 0$$

$x-5=0$ or $x-2=0$

$x=5$ $\quad$ $x=2$

When you substitute $x=2$ into the original equation, $x-3$ becomes -1 and $x-4$ becomes -2. The log of a negative number cannot be determined. Therefore $x=2$ does not satisfy the equation. **(3 marks)**

4. Find the value of k.

Substitute $N(t)=100$ and $t=2$ into the equation.

$$N(t)=k(2)^{\frac{t}{2}}$$
$$100=k(2)^{\frac{2}{2}}$$
$$k=50$$

$$\therefore N(t)=50(2)^{\frac{t}{2}}$$

Substitute $t=14$ h into the equation.

$$N(t)=50(2)^{\frac{14}{2}}$$
$$=6400$$

There are 6400 bacteria present after 14 h. **(3 marks)**

5. $$\frac{\log_5 625}{\log_{16}\left[(4)\left(\sqrt{8}\right)\right]}=\frac{\log_5 5^4}{\log_{16} 4+\log_{16}\sqrt{8}}$$
$$=\frac{4}{\log_{16} 4+\log_{16}\sqrt{8}}$$

Let $\log_{16} 4=x$ and $\log_{16}\sqrt{8}=y$.

$$4=16^x$$
$$4=4^{2x}$$
$$2x=1$$
$$x=\frac{1}{2}$$

$$8^{\frac{1}{2}}=16^y$$
$$2^{\frac{3}{2}}=2^{4y}$$
$$4y=\frac{3}{2}$$
$$y=\frac{3}{8}$$

$$\therefore \frac{\log_5 625}{\log_{16}(4)\left(\sqrt{8}\right)}=\frac{4}{\left(\frac{1}{2}\right)+\left(\frac{3}{8}\right)}$$
$$=\frac{4}{\frac{7}{8}}$$
$$=\frac{32}{7}$$ **(5 marks)**

6. $\log_{(x+y)} 16 = 2$

$16 = (x+y)^2$

$4^2 = (x+y)^2$

$x+y=4$ (1)

$x+y=4$ (1)

$2x-y=-2$ (2)

(1) + (2): $3x=2$

$x=\frac{2}{3}$

Substitute $x=\frac{2}{3}$ in (1).

$\frac{2}{3}+y=4$

$y=3\frac{1}{3}$

$(x,y)=\left(\frac{2}{3},3\frac{1}{3}\right)$ **(3 marks)**

$3^{2x-y}=9^{-1}$

$3^{2x-y}=3^{-2}$

$2x-y=-2$ (2)

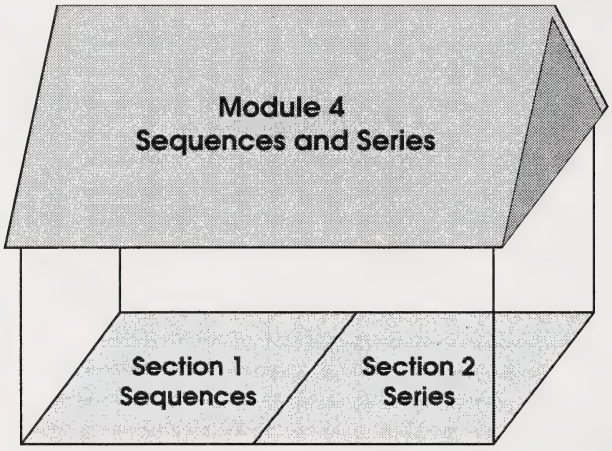
Module 4: Sequences and Series

Overview

In studying sequences and series, students should come to understand the power of sequences and series when describing recurrence relations. That is, formulas expressing each term as a function of one or more of the previous terms. Students should understand the role of recurrence formulas for solving enumeration problems, since these can be translated easily to computer programs to obtain solutions. Students should use difference-equation techniques to express recurrence relations in closed form; that is, the nth term written as a function of n.

Recurrence relations can be used to model real-world phenomena.

The study of finite sequences and series leads to consideration of the corresponding infinite cases and concepts associated with limiting processes.



Evaluation

The evaluation of this module is based on two section assignments and a final module assignment:

Section 1 Assignment	**40 marks**
Section 2 Assignment	**30 marks**
Final Module Assignment	**30 marks**
TOTAL	**100 marks**

Optional Videos

Sections 1 and 2: Extra Help

Arithmetic Sequences and Series, *Catch 30* series, ACCESS Network, available from the Learning Resources Distributing Centre.
Geometric Sequences and Series, *Catch 30* series, ACCESS Network, available from the Learning Resources Distributing Centre.

Section 1: Sequences

Key Concepts

- identifying sequences
- investigating patterns in sequences
- using the recursive definition
- investigating arithmetic sequences
- applying the general term formula for arithmetic sequences
- investigating geometric sequences
- applying the general term formula for geometric sequences

Section 1: Assignment Answer Key (40 marks)

1. $t_1 = 2(1)^2 - 3(1)$ $t_2 = 2(2)^2 - 3(2)$ $t_3 = 2(3)^2 - 3(3)$ $t_4 = 2(4)^2 - 3(4)$

$= 2 - 3$ $= 2(4) - 6$ $= 18 - 9$ $= 32 - 12$

$= -1$ $= 2$ $= 9$ $= 20$

The first four terms are -1, 2, 9, and 20. **(2 marks)**

2. $t_1 = 3$

$t_2 = 7$ or $t_2 = t_1 + 4$

$t_3 = 11$ or $t_3 = t_2 + 4$

The nth term is 4 more than the $(n-1)$ term.

Therefore, the recursion formula is as follows:

$t_n = t_{n-1} + 4$, where $n > 1$ and $t_1 = 3$ or $t_{n+1} = t_n + 4$, where $t_1 = 3$ **(3 marks)**

3. $t_n = -13$, $a = 20$, and $d = 17 - 20$

$= -3$

$t_n = a + (n-1)d$

$-13 = 20 + (n-1)(-3)$

$-13 = -3n + 23$

$3n = 36$

$n = 12$

The 12th term is -13. **(2 marks)**

4. $a=-16,\ n=14$, and $d=-12-(-16)$
$=4$

$t_n=a+(n-1)d$
$t_{14}=-16+(14-1)(4)$
$=-16+(13)(4)$
$=-16+52$
$=36$ **(1 mark)**

$t_n=a+(n-1)d$
$=-16+(n-1)4$
$=-16+4n-4$
$=4n-20$ **(1 mark)**

5. $t_n=30,\ a=-2$, and $n=2+3$
$=5$

$t_n=a+(n-1)d$
$30=-2+(5-1)d$
$32=4d$
$d=8$

$t_2=t_1+8$
$=-2+8$
$=6$

$t_3=t_2+8$
$=6+8$
$=14$

$t_4=t_3+8$
$=14+8$
$=22$

The three arithmetic means are 6, 14, and 22. **(2 marks)**

6. $t_6=-12$ and $n=6$

$t_n=a+(n-1)d$
$-12=a+(6-1)d$
$-12=a+5d$ (1)

$t_{20}=-96$ and $n=20$

$t_n=a+(n-1)d$
$-96=a+(20-1)d$
$-96=a+19d$ (2)

Subtract (1) from (2).

$-96=a+19d$
$-12=a+5d$
$-84=14d$
$-6=d$

Substitute $d=-6$ in (1).

$-12=a+5(-6)$
$-12=a-30$
$a=18$

$t_n=a+(n-1)d$
$=18+(n-1)(-6)$
$=18-6n+6$
$=-6n+24$ **(4 marks)**

7. $a = 10,\ d = 3,$ and $n = 25$

$$\begin{aligned} t_n &= a + (n-1)d \\ t_{25} &= 10 + (25-1)3 \\ &= 10 + 72 \\ &= 82 \end{aligned}$$

Ismail spends 82 min riding his bike. **(3 marks)**

8. $a = \$25\,000,\ t_n = 49\,000,$ and $d = \$1200$

$$\begin{aligned} t_n &= a + (n-1)d \\ 49\,000 &= 25\,000 + (n-1)1200 \\ 49\,000 &= 25\,000 + 1200n - 1200 \\ 49\,000 &= 23\,800 + 1200n \\ 25\,200 &= 1200n \\ n &= 21 \end{aligned}$$

It will take Carlotta 21 years to earn \$49 000 per year. **(3 marks)**

9.
$$\begin{aligned} r &= \frac{t_3}{t_2} \\ &= \frac{2}{2\sqrt{2}} \\ &= \frac{1}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2} \end{aligned}$$

$$\begin{aligned} t_4 &= t_3 \times r \\ &= 2 \times \frac{\sqrt{2}}{2} \\ &= \sqrt{2} \end{aligned}$$

$$\begin{aligned} t_5 &= t_4 \times r \\ &= \sqrt{2} \times \frac{\sqrt{2}}{2} \\ &= \frac{2}{2} \\ &= 1 \end{aligned}$$

The common ratio is $\frac{\sqrt{2}}{2}$, and $t_4 = \sqrt{2}$ and $t_5 = 1$. **(3 marks)**

10. $a = -6$ and $r = \dfrac{4}{-6}$

$= -\dfrac{2}{3}$

$t_n = ar^{n-1}$

$t_n = (-6)\left(-\dfrac{2}{3}\right)^{n-1}$

$t_9 = (-6)\left(-\dfrac{2}{3}\right)^{9-1}$

$= (-6)\left(-\dfrac{2}{3}\right)^{8}$

$= \left(\dfrac{2}{-\cancel{6}}\right)\left(\dfrac{256}{\cancel{6561}\ 2187}\right)$

$= -\dfrac{512}{2187}$ **(3 marks)**

11. $a = -60$ and $r = \dfrac{120}{-60}$

$= -2$

$t_n = ar^{n-1}$

$-3840 = (-60)(-2)^{n-1}$

$64 = (-2)^{n-1}$

$(-2)^6 = (-2)^{n-1}$

$6 = n - 1$

$n = 7$

$t_n = ar^{n-1}$

$30\,720 = (-60)(-2)^{n-1}$

$-512 = (-2)^{n-1}$

$(-2)^9 = (-2)^{n-1}$

$9 = n - 1$

$n = 10$

The seventh term has a value of -3840, and there are a total of 10 terms in the entire sequence. **(4 marks)**

12. $t_7 = 1792$ and $n = 7$

$$t_n = ar^{n-1}$$
$$t_7 = ar^{7-1}$$
$$1792 = ar^6 \quad (1)$$

$t_{10} = -14\,336$ and $n = 10$

$$t_n = ar^{n-1}$$
$$t_{10} = ar^{10-1}$$
$$-14\,336 = ar^9 \quad (2)$$

Divide (2) by (1).

$$\frac{ar^9}{ar^6} = \frac{-14\,336}{1792}$$
$$r^3 = -8$$
$$r = -2$$

Substitute $r = -2$ in (1).

$$1792 = ar^6$$
$$1792 = a(-2)^6$$
$$1792 = a(64)$$
$$28 = a$$

$$t_n = ar^{n-1}$$
$$t_n = (28)(-2)^{n-1}$$

Thus, $r = -2$, $a = 28$, and $t_n = (28)(-2)^{n-1}$. **(3 marks)**

13. **a.** $A = P(1+i)^n$

$$= 1800(1.02)^{20}$$
$$= 1800(1.485\,947\,396)$$
$$= \$2674.71$$

After five years, Valentino has \$2674.71. **(1 mark)**

b. $I = A - P$

$$= 2674.71 - 1800$$
$$= \$874.71$$

Valentino earned \$874.71 in interest. **(1 mark)**

c. $A = 2400$, $P = 1800$, and $i = 0.02$

The variable n represents the number of quarter-year intervals.

$$A = P(1+i)^n$$
$$2400 = 1800(1.02)^n$$
$$24 = 18(1.02)^n$$
$$\log 24 = \log\left[18(1.02)^n\right]$$
$$\log 24 = \log 18 + n \log 1.02$$
$$n = \frac{\log 24 - \log 18}{\log 1.02}$$
$$\doteq 14.527$$

Thus, 15 quarter-year intervals are required, since no interest will be paid for a fraction of a quarter year. Valentino has to wait $\frac{15}{4} = 3\frac{3}{4}$ years. **(2 marks)**

d. $A = \$5200$, $n = 12$, and $i = \dfrac{12\%}{2}$

$= 6\%$

$= 0.06$

$P_v = A(1+i)^{-n}$

$= 5200(1.06)^{-12}$

$= \$2584.24$

At 12% per annum compounded semiannually, \$2584.24 should be invested now in order to produce \$5200 in six years. **(2 marks)**

Section 2: Series

Key Concepts

- identifying series
- using sigma notation
- determining the sum of a finite series
- deriving the sum formula for an arithmetic series
- applying the sum formula of an arithmetic series
- deriving the sum formula for a geometric series
- applying the sum formula for a geometric series

Section 2: Assignment Answer Key (30 marks)

1. $a = \dfrac{1}{8}$ and $r = \dfrac{\frac{1}{4}}{\frac{1}{8}}$

$= 2$

$t_n = ar^{n-1}$

$= \left(\dfrac{1}{8}\right)(2)^{n-1}$ or 2^{n-4}

The number of terms must be determined.

$32 = \left(\dfrac{1}{8}\right)(2)^{n-1}$

$256 = 2^{n-1}$

$2^8 = 2^{n-1}$

$n = 9$

$\therefore \sum_{n=1}^{9} \frac{1}{8}(2)^{n-1}$ or $\sum_{n=1}^{9}(2)^{n-4}$ **(3 marks)**

2. $\sum_{n=3}^{8} n^2 - 8n = \left[3^2 - 8(3)\right] + \left[4^2 - 8(4)\right] + \left[5^2 - 8(5)\right] + \left[6^2 - 8(6)\right] + \left[7^2 - 8(7)\right] + \left[8^2 - 8(8)\right]$

$= [9-24] + [16-32] + [25-40] + [36-48] + [49-56] + [64-64]$

$= [-15] + [-16] + [-15] + [-12] + [-7] + [0]$

$= -65$ **(2 marks)**

3. $a = 26,\ n = 10,$ and $d = 22 - 26$

$= -4$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{10} = \frac{10}{2}[2(26) + (10-1)(-4)]$

$= 5[52 - 36]$

$= 5(16)$

$= 80$ **(2 marks)**

4. $t_n = 84,\ a = -30,$ and $d = -24 - (-30)$

$= 6$

The number of terms must be determined.

$t_n = a + (n-1)d$

$84 = -30 + (n-1)(6)$

$84 = -30 + 6n - 6$

$120 = 6n$

$n = 20$

$S_n = \frac{n}{2}(a + t_n)$

$S_{20} = \frac{20}{2}(-30 + 84)$

$= 10(54)$

$= 540$

The sum of the arithmetic series is 540. **(3 marks)**

5. $S_{16} = -80$ and $n = 16$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$-80 = \frac{16}{2}[2a + (16-1)d]$

$-80 = 8[2a + 15d]$

$-10 = 2a + 15d$ (1)

$S_{22} = -506$ and $n = 22$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$-506 = \frac{22}{2}[2a + (22-1)d]$

$-506 = 11[2a + 21d]$

$-46 = 2a + 21d$ (2)

Subtract (1) from (2).

$$\begin{aligned} -46 &= 2a + 21d \\ -10 &= 2a + 15d \\ \hline -36 &= 6d \\ d &= -6 \end{aligned}$$

Substitute $d = -6$ in (1).

$$\begin{aligned} -10 &= 2a + 15d \\ -10 &= 2a + 15(-6) \\ 80 &= 2a \\ a &= 40 \end{aligned}$$

$$\begin{aligned} S_n &= \frac{n}{2}[2a + (n-1)d] \\ S_{12} &= \frac{12}{2}[2(40) + (11)(-6)] \\ &= 6(80 - 66) \\ &= 6(14) \\ &= 84 \end{aligned}$$

The common difference is -6; the first term is 40; and the sum of the first twelve terms is 84. **(4 marks)**

6. $a = 800$, $d = 75$, and $n = 6$

$$\begin{aligned} S_n &= \frac{n}{2}[2a + (n-1)d] \\ &= \frac{6}{2}[2(800) + (6-1)75] \\ &= 3[1600 + 375] \\ &= 5925 \end{aligned}$$

The first job pays \$5925 for six months.

$a = 100$, $d = 25$, and $n = 24$

$$\begin{aligned} S_n &= \frac{n}{2}[2a + (n-1)d] \\ &= \frac{24}{2}[2(100) + (24-1)25] \\ &= 12[200 + 575] \\ &= 9300 \end{aligned}$$

The second job pays \$9300 for six months.

Therefore, Kecheni should accept the second job. **(4 marks)**

7. $S_n = 1700,\ a = 80,$ and $d = 20$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$
$$1700 = \frac{n}{2}\left[2(80) + (n-1)(20)\right]$$
$$1700 = \frac{n}{2}(160 + 20n - 20)$$
$$1700 = \frac{n}{2}(140 + 20n)$$
$$3400 = 140n + 20n^2$$
$$20n^2 + 140n - 3400 = 0$$
$$n^2 + 7n - 170 = 0$$
$$(n+17)(n-10) = 0$$

$n + 17 = 0$ or $n - 10 = 0$

$n = -17$ $\quad$ $n = 10$

Since $n = -17$ is not a whole number, then it takes 10 months to pay off the \$1700. **(4 marks)**

8. $a = 1800,\ r = \frac{1}{2},$ and $n = 10$

$$S_n = \frac{a\left(r^n - 1\right)}{r - 1}$$
$$= \frac{1800\left[\left(\frac{1}{2}\right)^{10} - 1\right]}{\frac{1}{2} - 1}$$
$$= \frac{1800\left[\frac{1}{1024} - 1\right]}{-\frac{1}{2}}$$
$$\doteq 3596.4843$$

Approximately 3596 L of milk is stored in the storage tank. **(4 marks)**

9. $S_6 = -168$, $S_7 = 344$, and $t_5 = 128$

$$S_6 + t_7 = S_7$$
$$-168 + ar^6 = 344 \qquad \left(t_7 = ar^6\right)$$
$$ar^6 = 512$$

$$t_5 = ar^4 = 128$$

Divide ar^6 by ar^4.

$$\frac{ar^6}{ar^4} = \frac{512}{128}$$
$$r^2 = 4$$
$$r = -2 \qquad (2 \text{ will not work.})$$

Solve for a.

$$ar^6 = 512$$
$$a(-2)^6 = 512$$
$$a = 8$$

Therefore, the first four terms are 8, -16, 32, and -64.

$$S_n = \frac{a\left(r^n - 1\right)}{r-1}$$
$$S_{12} = \frac{8\left[(-2)^{12} - 1\right]}{-2-1}$$
$$= \frac{8(4096-1)}{-3}$$
$$= -10\,920 \quad \textbf{(4 marks)}$$

Final Module Assignment Answer Key (30 marks)

Part A: Multiple Choice

Each answer in this part is worth 1 mark.

1. D **2.** C **3.** B **4.** B **5.** C **6.** A **7.** C **8.** D
(8 marks)

Part B: Numerical Response

Each answer in this part is worth 1 mark.

1. 0.39 **2.** 3280 **3.** 58 **4.** 12.0
(4 marks)

Part C: Written Response

1. a. Graph A shows a geometric series.

The ordered pairs for Graph A are as follows:

$$(1,1),\ (2,2),\ (3,4),\ (4,8)$$

Therefore, the series is $1+2+4+8$.

Since $r=2$, then the graph is a geometric series. **(2 marks)**

b. The ordered pairs for Graph B are as follows:

$$(1,0),\ (2,1),\ (3,2),\ (4,3),\ (5,4),\ (6,5),\text{ and }(7,6)$$

Therefore, the series is $0+1+2+3+4+5+6$.

Since $d=1$, then Graph B is an arithmetic series. **(2 marks)**

c. The graph of a geometric series is a curve. The graph of an arithmetic series is linear. **(1 mark)**

2. Let $t_2 = x$. Thus, $t_3 = 42-(x+2)$.

Thus, the sequence is $2,\ x,\ 42-(x+2)$.

$$r = \frac{x}{2} = \frac{42-x-2}{x}$$
$$x^2 = 80-2x$$
$$x^2+2x-80=0$$
$$(x+10)(x-8)=0$$
$$x = 8 \quad (\text{since } x=-10 \text{ is inadmissable})$$

$t_2 = 8,\ n=5,$ and $r = \frac{8}{2}$

$= 4$

$$S_n = \frac{a\left(r^n-1\right)}{r-1}$$
$$S_5 = \frac{2\left[(4)^5-1\right]}{4-1}$$
$$= \frac{2(1023)}{3}$$
$$= 682 \quad \textbf{(5 marks)}$$

3. $t_1 = 3$

$t_2 = t_1 + t_1 = 6$

$t_3 = t_2 + t_1 = 9$

$t_4 = t_3 + t_1 = 12$

$a = 3,\ d = 3,$ and $n = 12$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

$$S_{12} = \frac{12}{2}\left[2(3) + (12-1)3\right]$$

$$= 6(39)$$

$$= 234$$ **(4 marks)**

4. The series for Plan A is as follows:

$25 + 30 + 35 + \ldots$

$a = 25,\ d = 5,$ and $n = 16$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

$$S_{16} = \frac{16}{2}\left[2(25) + (16-1)5\right]$$

$$= 8(50 + 75)$$

$$= \$10.00$$

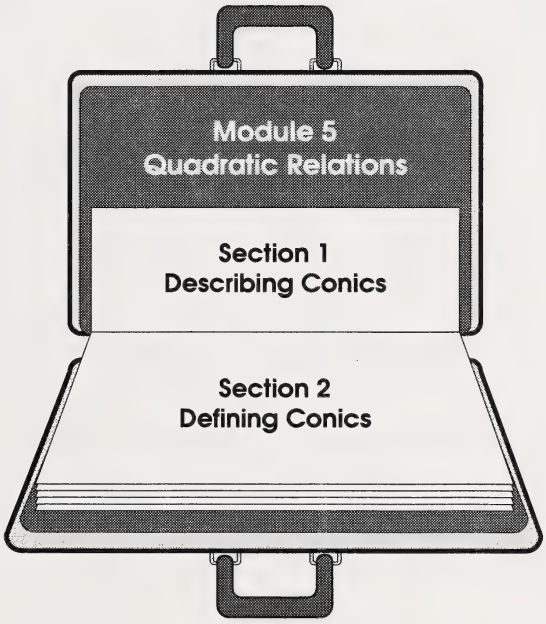
Therefore, Plan B is better because it is \$1.00 cheaper. **(4 marks)**

Module 5: Quadratic Relations

Overview

The study of quadratic relations connects several sections from the high school mathematics program. Coordinate geometry and algebra are linked to the analysis of relations and the resulting graphs of the locus of these relations. Students can see the connection between real-world applications and theoretical representations because of a number of practical phenomena that can be described in terms of quadratic relations.

Conics have many applications in science and technology. You can design navigational systems, telescopes, antennae, car headlights, and long-range microphones.



Evaluation

The evaluation of this module is based on two assignments and a final module assignment.

Section 1 Assignment	**38 marks**
Section 2 Assignment	**36 marks**
Final Module Assignment	**26 marks**
TOTAL	**100 marks**

Materials and Equipment

Section 1: Activity 1

- clay, pasticine
- coloured solution
- water
- styrofoam
- cone-shaped cup
- cylindrical pipe
- flashlight
- car tire
- rubber sheet

Section 2: Activity 1

- mathematical set
- string
- cardboard
- sheet of paper
- pencil

Section 2: Activity 2

- double-concentric-circle graph paper
- circle-line graph paper

Mandatory Videos

Section 1: Activity 1

Cutting the Curves, *Discovering Conics* series, ACCESS Network, available from the Learning Resources Distributing Centre.

Section 1: Activity 2

Graphing the Curves, *Discovering Conics* series, ACCESS Network, available from the Learning Resources Distributing Centre.

Section 2: Activity 1

Defining the Locus, *Discovering Conics* series, ACCESS Network, available from the Learning Resources Distributing Centre.

Section 2: Activity 2

Eccentricity, *Discovering Conics* series, ACCESS Network, available from the Learning Resources Distributing Centre.

Section 1: Describing Conics

Key Concepts

- determining physical properties of the conic sections with respect to the intersection of a plane and a double-napped cone
- describing the effects of changing the numerical coefficients in the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $B = 0$

Section 1: Assignment Answer Key (38 marks)

1. **a.** The degenerate conic sections are a point, a line, two parallel lines, two intersecting lines, and the case of no locus. **(1 mark)**
b. A line, a point, and two intersecting lines can be demonstrated. **(1 mark)**

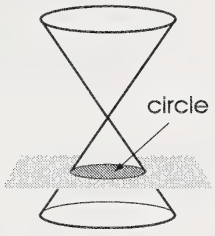
2. The two focal points of the ellipse approach each other until they coincide. **(1 mark)**

3. The typical conic sections are a parabola, an ellipse, a hyperbola, and a circle. **(1 mark)**

4. In order to get a hyperbola, you should cut a double-napped cone so that both nappes are intersected. The angle of the cutting plane must exceed the angle between a generator and the horizontal. If you cut through the common vertex, you will get the degenerate case of the two intersecting lines. **(1 mark)**

5. When you rotate a right triangle about its vertical arm, you get a right circular cone. **(1 mark)**

6. The intersection of a horizontal plane and a double-napped cone is a circle. **(1 mark)**



7. The first person to use the double-napped cones and an intersecting plane to show the conic sections was Appolonius, a Greek mathematician. He performed the demonstration around 200 B.C. in Greece. **(2 marks)**

8. D **(1 mark)**

9. B **(1 mark)**

10. D **(1 mark)**

11. D **(1 mark)**

12. B **(1 mark)**

13. **a.** This equation would produce a hyperbola. $\left(\frac{1}{2}\ \textbf{mark}\right)$

    b. This equation would produce an ellipse. $\left(\frac{1}{2}\ \textbf{mark}\right)$

    c. This equation would produce a circle. $\left(\frac{1}{2}\ \textbf{mark}\right)$

    d. This equation would produce a parabola. $\left(\frac{1}{2}\ \textbf{mark}\right)$

14. **a.** The parabola opens in an upward direction. $\left(\frac{1}{2}\ \textbf{mark}\right)$

    b. The vertex is in the third quadrant. $\left(\frac{1}{2}\ \textbf{mark}\right)$

    c. The graph would get narrower. $\left(\frac{1}{2}\ \textbf{mark}\right)$

    d. The graph would open in a downward direction. $\left(\frac{1}{2}\ \textbf{mark}\right)$

15. **a.** The equation represents an ellipse. $\left(\frac{1}{2}\ \textbf{mark}\right)$

b. The centre is located in the second quadrant. $\left(\frac{1}{2}\ \textbf{mark}\right)$

c. The centre would be located in the third quadrant. $\left(\frac{1}{2}\ \textbf{mark}\right)$

d. The curve would be larger. $\left(\frac{1}{2}\ \textbf{mark}\right)$

16. **a.** The equation represents a hyperbola. **(1 mark)**
b. There would be two intersecting lines through the origin. **(1 mark)**

17. **a.** Answers will vary. One possibility is $A=7,\ C=2,\ D=16,\ E=-8,\ F=-81$. **(1 mark)**

b. The following parameters determine the shape: $A \neq C;\ A>0,\ C>0$. **(1 mark)**

c. The following parameters determine the position: $D>0$ and $E<0$ since the centre is in the second quadrant. $A>C$ since the longer axis of the ellipse is located parallel to the y-axis. **(1 mark)**

18. **a.** Answers will vary. One possibility is $A=1,\ C=-4,\ D=7,\ E=-9,\ F=-16$. **(1 mark)**

b. The following parameters determine the shape: A and C have opposite signs. **(1 mark)**

c. The following parameters determine the position:

- $A>0,\ C<0$, and $F<0$ since the graph is along the x-axis.
- $D>0$ since the graph is shifted to the left. Negative E moves the hyperbola along the x-axis downward. **(1 mark)**

19. The value of $C=9$ would make the ellipse a circle. **(1 mark)**

20. As the ellipse becomes more and more round, the values of A and C get closer to the same value. When the curve is a circle, the values of A and C are equal. **(1 mark)**

21. **a.** The graph moves to the right. **(1 mark)**
b. The graph becomes larger. **(1 mark)**

22. **a.** The equation represents an ellipse. **(1 mark)**

b. The ellipse rotates 90°. In other words, the graph becomes an ellipse with its longer axis parallel to the y-axis. **(1 mark)**

c. For an ellipse, negative D values move the graph to the right and positive E values move the graph down. This would put the centre in the fourth quadrant. **(1 mark)**

23. Start with an equation of a circle, like $x^2+y^2+F=0$. Since the curve is a circle, you need $A=C$. To put the circle on the crossing lines or origin, you need D and E equal to zero. A small F value will give a small circle for the button. Using $F=-0.25$ will give a circle with a diameter of 1 unit which represents 1 m. The next circle has a diameter of 4 m so you need a radius of 2 m. Using $F=-4$ will give a circle with a radius of 2 m. The next circle has a diameter of 8 m or a radius of 4 m so you use $F=-16$ to get the circle with a radius of 4 m. The largest circle has a diameter of 12 m or a radius of 6 m, so you need an F value of -36. **(3 marks)**

Section 2: Defining Conics

Key Concepts

- describing each conic as a locus of points
- using eccentricity to define conics

Section 2: Assignment Answer Key (36 marks)

1. When the distance from the focus to the directrix is increased, the parabola becomes wider. When the distance between the focus and the directrix is decreased, the parabola becomes narrower. **(1 mark)**

2. The statement fits the locus definition for a hyperbola. Therefore, the path of the comet is hyperbolic. **(1 mark)**

3. If one focus is fixed and the other moves away, the ellipse gets narrower.

If both foci are changed, the ellipse narrows and gets more elongated.

One restriction is that the distance between the foci must be less than the length of the string.
(2 marks)

4. By the locus definition for an ellipse, $d(PA)+d(PB)=k$. Using this formula and the distance formula $\left(d=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\right)$, the equation for this ellipse is determined. Note that k, the constant, must be given.
(2 marks)

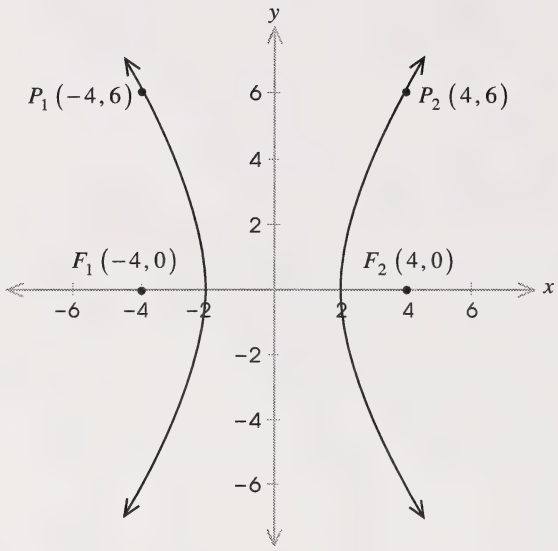
5. $\left|d(PF_1)-d(PF_2)\right|=3.2$ cm **(2 marks)**

6. Choose a value for x or y and substitute in $3x^2-y^2-12=0$ to find the other variable. Using these points and the focal points, find the constant.

Let $y=6$.

$$3x^2-(6)^2-12=0$$
$$3x^2-36-12=0$$
$$3x^2=48$$
$$x^2=16$$
$$x=\pm 4$$

The two points are $P_1(4,6)$ and $P_2(-4,6)$.



By the locus definition for a hyperbola, $\left|d(PF_1)-d(PF_2)\right|=k$.

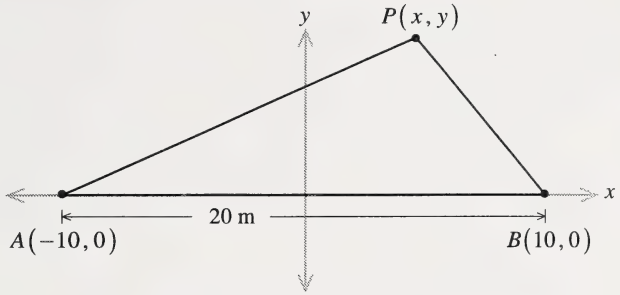
$$\begin{aligned}\left|d(P_1F_1)-d(P_1F_2)\right| &= \left|\sqrt{\left[4-(-4)^2\right]+(6-0)^2}-\sqrt{(4-4)^2+(6-0)^2}\right| \\ &= \left|\sqrt{(8)^2+(6)^2}-\sqrt{(0)^2+(6)^2}\right| \\ &= \left|\sqrt{64+36}-\sqrt{36}\right| \\ &= |10-6| \\ &= 4\end{aligned}$$

$$\begin{aligned}\left|d(P_2F_1)-d(P_2F_2)\right| &= \left|\sqrt{\left[-4-(-4)^2\right]+(6-0)^2}-\sqrt{(-4-4)^2+(6-0)^2}\right| \\ &= \left|\sqrt{(0)^2+(6)^2}-\sqrt{(-8)^2+(6)^2}\right| \\ &= \left|\sqrt{36}-\sqrt{64+36}\right| \\ &= |6-10| \\ &= 4\end{aligned}$$

Therefore, $\left|d(PF_1)-d(PF_2)\right|=4$ and the locus definition holds for the hyperbola. **(4 marks)**

7. a. An ellipse is formed. By the locus definition for an ellipse, $d(PA)+d(PB)=k$.

The coordinates of A and B are $(-10,0)$ and $(10,0)$ respectively.



$$d(PA)+d(PB)=40$$

$$\sqrt{[x-(-10)]^2+(y-0)^2}+\sqrt{(x-10)^2+(y-0)^2}=40$$

$$\sqrt{(x+10)^2+y^2}=40-\sqrt{(x-10)^2+y^2}$$

$$\left(\sqrt{x^2+20x+100+y^2}\right)^2=\left(40-\sqrt{x^2-20x+100+y^2}\right)^2$$

$$x^2+20x+100+y^2=1600-80\sqrt{x^2-20x+100+y^2}+x^2-20x+100+y^2$$

$$40x-1600=-80\sqrt{x^2-20x+100+y^2} \quad \text{(Divide by 40.)}$$

$$x-40=-2\sqrt{x^2-20x+100+y^2} \quad \text{(Square both sides.)}$$

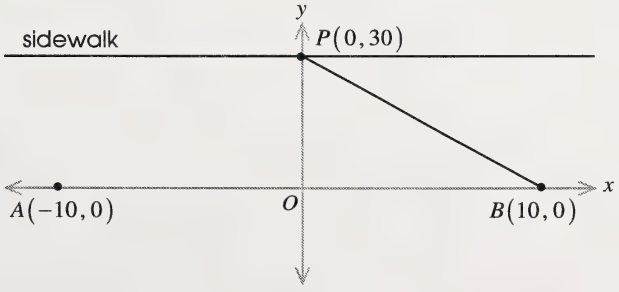
$$x^2-80x+1600=4\left(x^2-20x+100+y^2\right)$$

$$x^2-80x+1600=4x^2-80x+400+4y^2$$

$$3x^2+4y^2-1200=0$$

The equation of the ellipse is $3x^2+4y^2-1200=0$. **(4 marks)**

b.



P is the farthest point on the ellipse from $\overline{AB}$. Note that $d(PO)=30$ m and $d(OB)=10$ m.

Therefore, using the Pythagorean Theorem, the following may be stated:

$$\begin{aligned} d(PB) &= \sqrt{(30)^2 + (10)^2} \\ &= \sqrt{900 + 100} \\ &= \sqrt{1000} \\ &\doteq 31.62 \end{aligned}$$

Note that $d(AP) = d(BP)$.

$$\begin{aligned} \therefore\ d(AP) + d(BP) &\doteq 2(31.62) \\ &\doteq 63 \end{aligned}$$

Thus, the maximum length the rope could be is approximately 63 m. **(2 marks)**

8. If the directrix and focus remain the same, the hyperbola with a larger eccentricity would be wider and the two sections of the hyperbola would be closer together. **(1 mark)**

9. Circle: The limiting shape of an ellipse is a circle as the eccentricity approaches 0.
Parabola: The locus of points is a parabola when $e = 1$.
Ellipse: The locus of points is an ellipse when $0 < e < 1$.
Hyperbola: The locus of points is a hyperbola when $e > 1$.
(2 marks)

10. The eccentricity is 0.8. **(1 mark)**

11. **a.**

$$\frac{\sqrt{(x-3)^2 + (y+2)^2}}{x+2} = \frac{3}{4}$$

$$x^2 - 6x + 9 + y^2 + 4y + 4 = \frac{9}{16}\left(x^2 + 4x + 4\right)$$

$$16x^2 - 96x + 144 + 16y^2 + 64y + 64 = 9x^2 + 36x + 36$$

$$7x^2 + 16y^2 - 132x + 64y + 172 = 0$$

The equation is $7x^2 + 16y^2 - 132x + 64y + 172 = 0$. **(2 marks)**

b. To find the coordinates of the vertices of the ellipse, substitute $y=-2$ in the equation.

$$7x^2+16y^2-132x+64y+172=0$$
$$7x^2+16(-2)^2-132x+64(-2)+172=0$$
$$7x^2+64-132x-128+172=0$$
$$7x^2-132x+108=0$$
$$(7x-6)(x-18)=0$$

$7x-6=0$ or $x-18=0$

$x=\frac{6}{7}$ $\qquad$ $x=18$

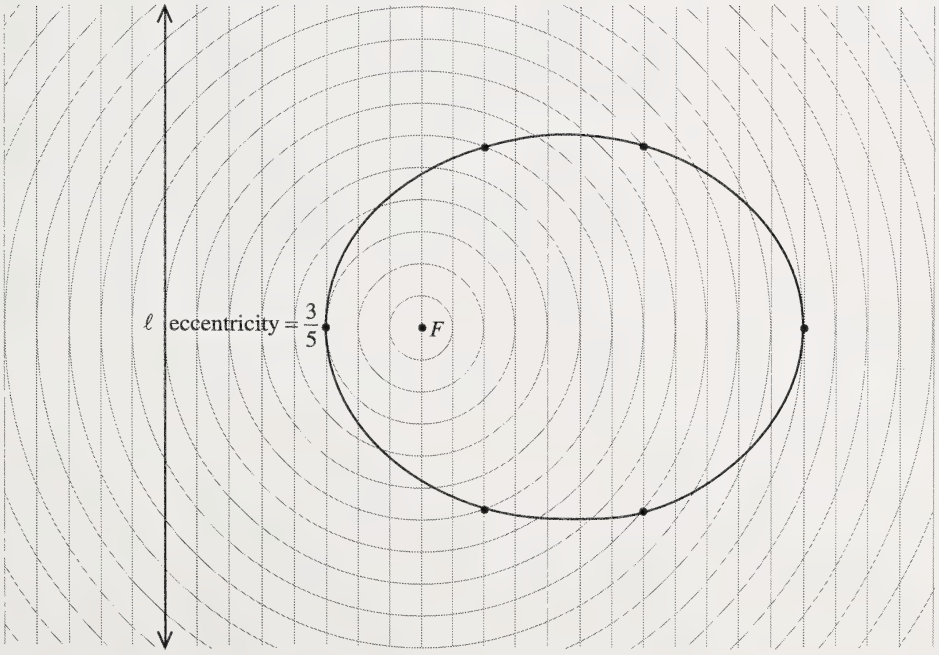
The coordinates of the vertices are $\left(\frac{6}{7},-2\right)$ and $(18,-2)$.

The distance between the vertex and the given focus is $3-\frac{6}{7}=2\frac{1}{7}$.

Therefore, the second focus is at $\left(18-2\frac{1}{7},-2\right)=\left(15\frac{6}{7},-2\right)$.

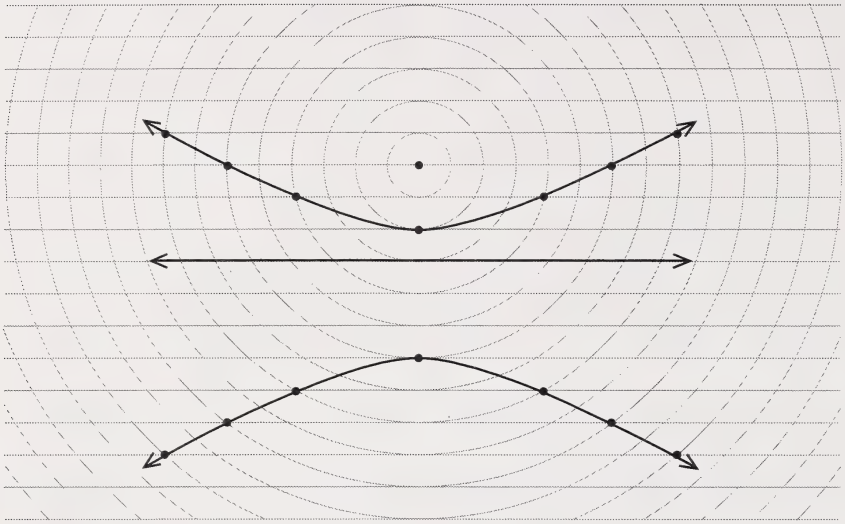
The listening posts are $\left(15\frac{6}{7}-3\right)=12\frac{6}{7}$ units apart. **(2 marks)**

12.



The conic is an ellipse. **(3 marks)**

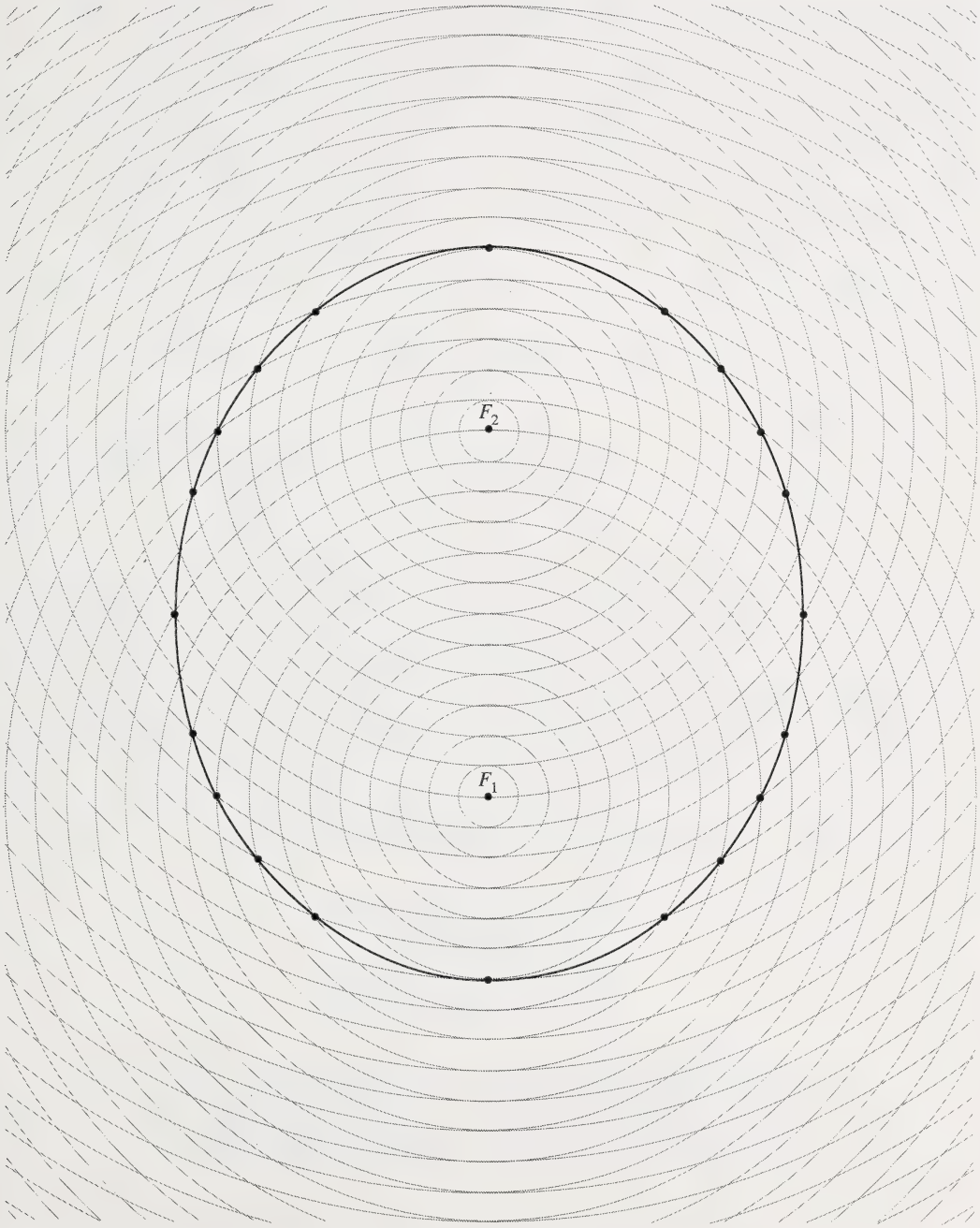
13.



(3 marks)

14. **C (1 mark)**

15.



(3 marks)

Final Module Assignment Answer Key (26 marks)

Part A: Multiple Choice

Each answer in this part is worth 1 mark.

1. A **2.** D **3.** B **4.** A **5.** C **6.** C **7.** D
(7 marks)

Part B: Numerical Response

Each answer in this part is worth 1 mark.

1. 1.75 **2.** 4.5 **3.** 90 **4.** 10
(4 marks)

Part C: Written Response

1. **a.** No, Jerome must use F_1 and ℓ_1 or F_2 and ℓ_2. He should choose a point on the curve and find the ratio of the distance to the focus and directrix. **(2 marks)**

b. By measurement, $e = 0.8$. **(1 mark)**

2. If A and B are foci and point P is a point on the curve, then the conic must be an ellipse because the only possibility is that the sum of the two distances from P to A and P to B is a constant.

If point Q is given, the conic can be an ellipse or a hyperbola. One cannot tell whether the sum or the difference of the distances from Q to A and Q to B is a constant. **(3 marks)**

3. The quadratic relation is a parabola with vertex $(0,0)$ because the directrix is a vertical line, the focus is on the x-axis, and the curve passes $(0,0)$ which is midpoint between the directrix and the focus. The distances from $(0,0)$ to $x=-3$ and the distance from $(0,0)$ to $(0,3)$ are equal. The distance from $(3,6)$ to $x=-3$ also equals the distance from $(3,6)$ to $(0,3)$. Therefore, the eccentricity must be 1 and the quadratic relation is a parabola. **(3 marks)**

4. Let $P(3,m)$ be the point on the graph. $F(2,2)$ is the focus and the directrix is D.

Since $e=1$, $PD=PF$.

$$\sqrt{(3-3)^2+(m-5)^2}=\sqrt{(3-2)^2+(m-2)^2}$$
$$m^2-10m+25=1+m^2-4m+4$$
$$-6m=-20$$
$$m=\frac{20}{6}$$
$$m=\frac{10}{3}$$ **(3 marks)**

5. $e = \frac{7}{5}$

$$\frac{\sqrt{(x-0)^2+(y-0)^2}}{5-x} = \frac{7}{5}$$

$$x^2 + y^2 = \frac{49}{25}\left(x^2 - 10x + 25\right)$$

$$25x^2 + 25y^2 = 49x^2 - 490x + 1225$$

$$24x^2 - 25y^2 - 490x + 1225 = 0$$

The equation of the conic is $24x^2 - 25y^2 - 490x + 1225 = 0$. **(3 marks)**

Module 6: Permutations, Combinations, and Statistics

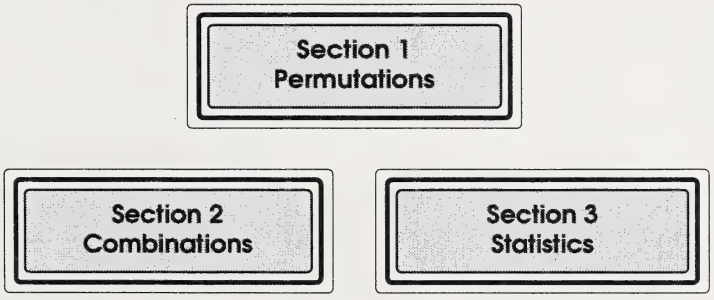
Overview

Permutations and combinations are important for processing, analysing, and communicating information.

Understanding of permutations and combinations is important for an understanding of the formal concepts of probability. This module emphasizes the development of a student's ability to solve problems using combinatorial analysis as opposed to the simple application of analytic formulas for permutations and combinations.

Statistics forms an essential part of the modern mathematics curriculum. Today, information is one of the most important commodities in society. The daily barrage of information must not only be understood; it must be used by students. Statistics begins with a problem that needs to be solved. It includes all the background thinking that occurs long before any work is done.

Module 6
Permutations, Combinations, and Statistics



Evaluation

The evaluation of this module is based on three assignments and a final module assignment:

Section 1 Assignment	**25 marks**
Section 2 Assignment	**28 marks**
Section 3 Assignment	**22 marks**
Final Module Assignment	**25 marks**
TOTAL	**100 marks**

Optional Videos

Sections 1 and 2: Extra Help

Permutations and Combinations, *Catch 30* series, ACCESS Network, available from the Learning Resources Distributing Centre.

Section 3: Extra Help

Graphing Statistics, *Graphing Mathematical Concepts* series, ACCESS Network, available from the Learning Resources Distributing Centre.

Statistics, *Catch 30* series, ACCESS Network, available from the Learning Resources Distributing Centre.

Section 1: Permutations

Key Concepts

- solving problems using the fundamental counting principle
- defining and using ${}_nP_n$ and $n!$
- defining and using ${}_nP_r$; solving linear permutations with repetitions, and permutations with like elements
- solving problems involving circular permutations and permutations with reflections
- solving probability questions involving the use of permutations

Section 1: Assignment Answer Key (25 marks)

1. A person can enter the theatre in seven ways and leave in seven ways. Therefore, the number of ways a person can enter and leave the theatre is $7 \times 7 = 49$ ways. **(1 mark)**

2.

2	10	10	5

The first digit can be 8 or 9. Therefore, there are two choices for the first digit.

The last digit has to be an odd number. There are five choices since 1, 3, 5, 7, or 9 can be used.

The second and third digits both can be any one of the ten digits; thus, there are ten choices each.

$$\therefore \text{ number of ways} = 2 \times 10 \times 10 \times 5$$
$$= 1000$$

A total of 1000 numbers can be formed. **(2 marks)**

3. Each letter can be mailed in three different ways (three mailboxes).

Therefore, four letters can be mailed in $3\times3\times3\times3=3^4=81$ ways. **(2 marks)**

4. a. $$\frac{(n-2)!}{(n-4)!2!}=\frac{(n-2)(n-3)\cancel{(n-4)!}}{\cancel{(n-4)!}2!}$$
$$=\frac{(n-2)(n-3)}{2}$$ **(2 marks)**

b. $$\frac{(n-3m-1)!3!}{(n-3m+1)!2!}=\frac{\cancel{(n-3m-1)!}(3)\cancel{(2)!}}{(n-3m+1)(n-3m)\cancel{(n-3m-1)!}\cancel{2!}}$$
$$=\frac{3}{(n-3m+1)(n-3m)}$$ **(2 marks)**

5. $8!=40\,320$

Eight different books can be arranged in 40 320 ways. **(1 mark)**

6. There are eleven books. Orasia can choose any seven of them and arrange them on the shelf.

$\therefore$ number of arrangements $= {}_{11}P_7$
$$=\frac{11!}{(11-7)!}$$
$$=11\times10\times9\times8\times7\times6\times5$$
$$=1\,663\,200$$

Orasia can arrange the books in 1 663 200 ways. **(2 marks)**

7. Peter can hang 1, 2, 3, or 4 pictures.

$\therefore$ number of ways $= {}_7P_1+{}_7P_2+{}_7P_3+{}_7P_4$
$$=\frac{7!}{6!}+\frac{7!}{5!}+\frac{7!}{4!}+\frac{7!}{3!}$$
$$=7+42+210+840$$
$$=1099$$

Peter can hang the pictures in 1099 ways. **(2 marks)**

8. There are five items taken all at a time, of which three are alike and two others are alike.

$\therefore$ number of ways $=\frac{5!}{3!2!}$
$$=\frac{5\times4}{2\times1}$$
$$=10$$

The items can be distributed in 10 ways. **(2 marks)**

9. If the English books stand together, there are $5!$ ways to arrange them.

If the French books stand together, there are $4!$ ways to arrange them.

The English books and the French books represent two different groups of books; thus, they can be arranged in $2!$ ways.

Therefore, Raschid can arrange these books in $5! \times 4! \times 2 = 5760$ ways. **(2 marks)**

10. Two marbles must be placed together. They can be considered as one group. These two marbles can be arranged in $2!$ ways.

The remaining thirteen marbles together with the one group of two marbles can be arranged in $(14-1)!$ ways.

$$\therefore \text{ number of arrangements} = 2!(14-1)!$$
$$= 2 \times 13!$$

The marbles can be arranged in $2 \times 13!$ ways. **(3 marks)**

11.
$$\text{number of ways} = \frac{(11-1)!}{2}$$
$$= \frac{10!}{2}$$
$$= 1\,814\,400$$

Eleven keys can be strung in 1 814 400 ways. **(1 mark)**

12. There are two red kings in a deck of 52 cards. They can be arranged in $2!$ ways.

If any two cards are chosen, the number of arrangements is ${}_{52}P_2$.

$$P = \frac{2!}{{}_{52}P_2}$$
$$= \frac{2!}{\left(\frac{52!}{50!}\right)}$$
$$= \frac{2!}{52 \times 51}$$
$$= \frac{1}{1326}$$

The probability that the two cards are red kings is $\frac{1}{1326}$. **(3 marks)**

Section 2: Combinations

Key Concepts

- defining and using applications of the formula ${}_nC_r = \frac{n!}{(n-r)!r!}$
- solving probability questions involving the use of combinations
- defining and using the Binomial Theorem and Pascal's triangle

Section 2: Assignment Answer Key (28 marks)

1. $$\frac{\frac{5!}{3!2!}}{\frac{5!}{2!3!}}+\frac{3!}{1!2!}\times\frac{\frac{3!}{2!1!}}{\frac{4!}{4!0!}}-\frac{7!}{7!0!}=1+3\times\frac{3}{1}-1$$

$$=9 \quad \textbf{(2 marks)}$$

2. You can select 7, 8, 9, or 10 candies.

$$\therefore \text{ number of ways} = {}_{10}C_7 + {}_{10}C_8 + {}_{10}C_9 + {}_{10}C_{10}$$
$$= \frac{10!}{7!3!}+\frac{10!}{8!2!}+\frac{10!}{9!1!}+\frac{10!}{10!0!}$$
$$= 120+45+10+1$$
$$= 176$$

Seven or more candies can be chosen in 176 ways. **(2 marks)**

3. Ten students can be selected from the twenty students in ${}_{20}C_{10}$ ways.

Each time you select a group of ten students, you get another group of ten students.

$$\therefore \text{ the number of ways} = \frac{{}_{20}C_{10}}{2}$$
$$= \frac{\frac{20!}{10!10!}}{2}$$
$$= \frac{20\times19\times18\times17\times16\times15\times14\times13\times12\times11}{10\times9\times8\times7\times6\times5\times4\times3\times2\times1\times2}$$
$$= 92\,378$$

The group can be separated in 92 378 ways. **(2 marks)**

4. $$\text{number of ways} = 2^6 - 1$$
$$= 63$$

George can choose the books in 63 ways. **(2 marks)**

5. Since the chairperson must be excluded and the secretary must be included, four students must be selected from the remaining eighteen students.

$$\therefore \text{ number of selections} = {}_1C_1 \times {}_{18}C_4$$
$$= 3060$$

The selection can be made in 3060 ways. **(2 marks)**

6. The number of ways in which any six flowers can be chosen is ${}_{35}C_6$.

The number of ways in which no rose is chosen is ${}_{20}C_6$.

The number of ways at least one rose is chosen is as follows:

$$\begin{aligned} {}_{35}C_6 - {}_{20}C_6 &= 1\,623\,160 - 38\,760 \\ &= 1\,584\,400 \end{aligned}$$

The flowers can be selected in 1 584 400 ways. **(2 marks)**

7. If no three of the nine points were collinear, the number of straight lines formed would be ${}_9C_2$.

If there were four noncollinear points, the number of straight lines formed by these four points would be ${}_4C_2$.

Four collinear points can form only one straight line.

The number of straight lines formed by joining any two of the nine points is as follows:

$$\begin{aligned} {}_9C_2 - {}_4C_2 + 1 &= \frac{9!}{7!2!} - \frac{4!}{2!2!} + 1 \\ &= 36 - 6 + 1 \\ &= 31 \end{aligned}$$

The number of lines formed is 31. **(2 marks)**

8. Find the number of ways in which four coins can be drawn.

$$\begin{aligned} {}_{14}C_4 &= \frac{14!}{10!4!} \\ &= 1001 \end{aligned}$$

Find the number of ways in which three quarters and one penny can be drawn.

$$\begin{aligned} {}_9C_3 \times {}_5C_1 &= \frac{9!}{6!3!} \times \frac{5!}{4!1!} \\ &= 84 \times 5 \\ &= 420 \end{aligned}$$

The required probability is $P = \frac{420}{1001}$. **(2 marks)**

9. There are four sour oranges and eight sweet oranges.

The number of ways to choose all five sweet oranges is ${}_8C_5$.

The number of ways to choose any five oranges is ${}_{12}C_5$.

The probability that all five oranges are sweet is $\frac{{}_8C_5}{{}_{12}C_5}$.

The probability that at least one orange is sour is as follows:

$$\begin{aligned}1-\frac{{}_8C_5}{{}_{12}C_5} &= 1-\frac{\frac{8!}{5!3!}}{\frac{12!}{5!7!}}\\ &= 1-\frac{56}{792}\\ &= \frac{736}{792}\\ &= \frac{92}{99}\end{aligned}$$

The probability that at least one orange is sour is $\frac{92}{99}$. **(3 marks)**

10. $\left(x^2-2y\right)^6 = \left(x^2\right)^6 - 6\left(x^2\right)^5(2y) + 15\left(x^2\right)^4(2y)^2 - 20\left(x^2\right)^3(2y)^3 + 15\left(x^2\right)^2(2y)^4 - 6\left(x^2\right)(2y)^5 + (2y)^6$

$= x^{12} - 12x^{10}y + 60x^8y^2 - 160x^6y^3 + 240x^4y^4 - 192x^2y^5 + 64y^6$ **(2 marks)**

11. $t_{k+1} = {}_nC_k x^{n-k} y^k$,

$t_{12} = t_{11+1}$

$= {}_{16}C_{11}\left(x^2\right)^{16-11}\left(-\frac{1}{2}\right)^{11}$

$= \frac{16!}{5!11!}\left(x^2\right)^5\left(-\frac{1}{2}\right)^{11}$

$= 4368x^{10}\left(\frac{-1}{2048}\right)$

$= -\frac{4368}{2048}x^{10}$ or $-2\frac{17}{128}x^{10}$

The twelfth term is $-\frac{4368}{2048}x^{10}$ or $-2\frac{17}{128}x^{10}$. **(2 marks)**

12. $t_{k+1} = {}_nC_k x^{n-k} y^k$

$t_7 = t_{6+1}$

$= {}_{11}C_6\left(x^2\right)^{11-6}(-3)^6$

$= \frac{11!}{5!6!}\left(x^2\right)^5(-3)^6$

$= 462x^{10}(729)$

$= 336\,798x^{10}$

The numerical coefficient is 336 798. **(2 marks)**

13. There are $5+6+7=18$ stamps.

Find the number of combinations of eighteen stamps taken two at a time.

$$
\begin{aligned}
{}_{18}C_2 &= \frac{18!}{2!16!} \\
&= \frac{18\times 17}{2} \\
&= 153
\end{aligned}
$$

Find the number of combinations of one stamp from Australia and one stamp from Japan.

$$
\begin{aligned}
{}_5C_1 \times {}_6C_1 &= 5\times 6 \\
&= 30
\end{aligned}
$$

Find the number of combinations of one stamp from Japan and one stamp from the United States.

$$
\begin{aligned}
{}_6C_1 \times {}_7C_1 &= 6\times 7 \\
&= 42
\end{aligned}
$$

$$
\begin{aligned}
P &= \frac{30}{153} + \frac{42}{153} \\
&= \frac{72}{153} \\
&= \frac{8}{17}
\end{aligned}
$$

The probability of drawing the stamps is $\frac{8}{17}$. **(3 marks)**

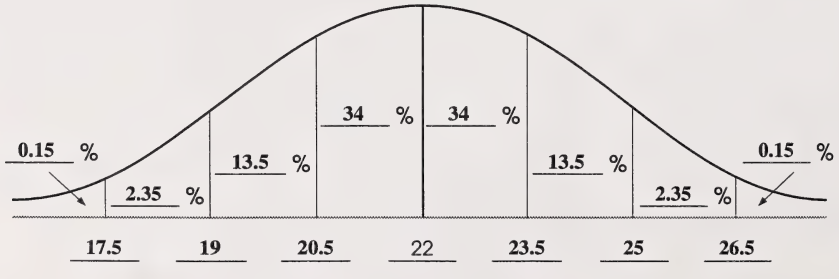
Section 3: Statistics

Key Concepts

- calculating and interpreting the mean and the standard deviation of a set of data
- identifying normal distribution
- using z-scores to solve situations that are normally distributed
- using z-scores to calculate the probability of an event happening

Section 3: Assignment Answer Key (22 marks)

1. a.



$\left(\frac{1}{2}\text{ mark}\right)$

b. $2.35 + 13.5 = 15.85$

The percentage of readings is 15.85%. $\left(\frac{1}{2}\ \textbf{mark}\right)$

c. $2(34 + 13.5) = 2(47.5)$
$= 95.0$

The percentage of data is 95.0%. $\left(\frac{1}{2}\ \textbf{mark}\right)$

d. $100 - (13.5 + 2.35 + 0.15) = 84$ or $34 + 34 + 13.5 + 2.35 + 0.15 = 84$

The percentage of data is 84%. $\left(\frac{1}{2}\ \textbf{mark}\right)$

2. a. $0.68 \times 8000 = 5440$

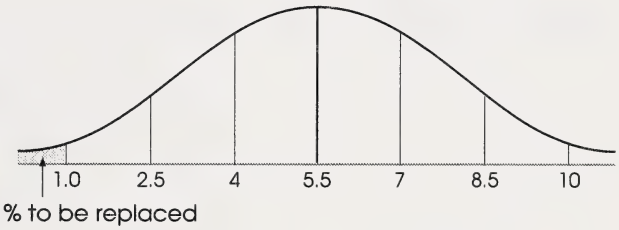
When 8000 boxes are produced, 5440 will be within 2 g of the required mass. **(1 mark)**

b. Using the normal distribution curve, $(2.5 + 2.5) = 5\%$ of the boxes have a mass between 176 g and 184 g; thus,

$\frac{5}{100} \times 8000 = 400$ boxes will not be accepted.

Four hundred boxes will not be accepted. **(1 mark)**

3. a. $z = \frac{x - \mu}{\sigma}$
$= \frac{1 - 5.5}{1.5}$
$= -3$



Area for $z = -3$ is 0.4987.

required area $= 0.5000 - 0.4987$
$= 0.0013$
$= 0.13\%$

The percentage of shears needing replacement is 0.13%. $\left(1\frac{1}{2}\ \textbf{marks}\right)$

b. $\frac{0.13}{100} \times 4000 = 5$

The number of shears to be replaced is 5. $\left(\frac{1}{2}\ \textbf{mark}\right)$

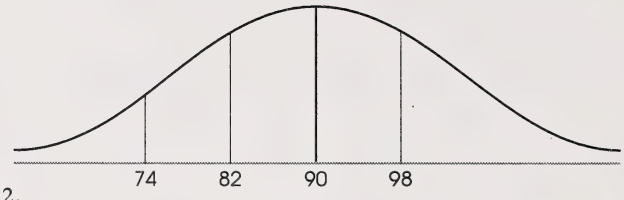
4. $z_1 = \dfrac{74-90}{8}$

$= -\dfrac{16}{8}$

$= -2$

$z_2 = \dfrac{98-90}{8}$

$= 1$



Use the z-score table to obtain the area from $z = 1$ to $z = -2$.

Area $= 0.4772 + 0.3413$

$= 0.8185$

$= 81.85\%$

The percentage of test scores is 81.85%. **(1 mark)**

5. ***z*-score for Math** ***z*-score for Physics**

$z = \dfrac{84-76}{12}$

$= 0.67$

$z = \dfrac{88-79}{16}$

$= \dfrac{9}{16}$

$= 0.56$

Sophia has a better relative standing in mathematics. **(2 marks)**

6. $z = \dfrac{x-\mu}{\sigma}$

$0.78 = \dfrac{x-67}{9}$

$x - 67 = 9(0.78)$

$x = 67 + 7.02$

$= 74$

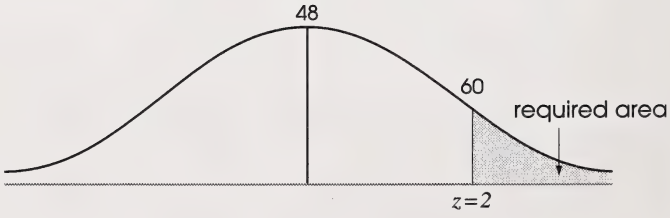
Bertram's test mark was 74%. **(1 mark)**

7. a. $z = \dfrac{x-\mu}{\sigma}$

$= \dfrac{60-48}{6}$

$= \dfrac{12}{6}$

$= 2$



The area for $z = 2$ is 0.4772.

The required area is $0.5 - 0.4772 = 0.0228$.

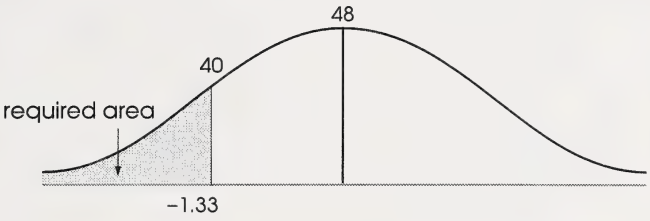
The probability that a battery lasts longer than sixty months is 0.0228. **(1 mark)**

b. $z = \dfrac{x-\mu}{\sigma}$

$= \dfrac{40-48}{6}$

$= -\dfrac{8}{6}$

$= -1.3\dot{3}$



The area for $z = -1.33$ is 0.4082.

The required area is $0.5 - 0.4082 = 0.0918$.

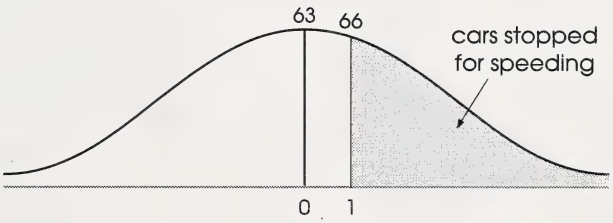
The percentage of batteries that will need to be replaced is 9.18%. **(1 mark)**

8. Since $60 + 0.10(60) = 60 + 6 = 66$, cars going 66 km/h or faster will be stopped for speeding.

$z = \dfrac{x-\mu}{\sigma}$

$= \dfrac{66-63}{3}$

$= 1$



The area for $z = 1$ is 0.3413.

required area $= 0.5000 - 0.3413$

$= 0.1587$

$0.1587 \times 120 = 19.044$

Therefore, nineteen cars are stopped for speeding. **(2 marks)**

9. $50\% - 3\% = 47\%$

z-score for $0.47 = -1.88$

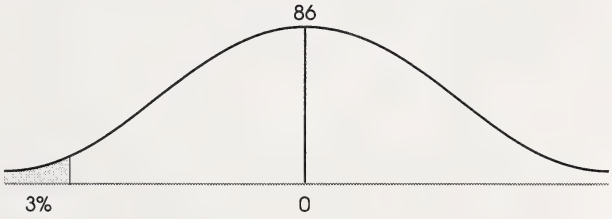
$z = \dfrac{x-\mu}{\sigma}$

$-1.88 = \dfrac{x-86}{36}$

$-67.68 = x - 86$

$86 - 67.7 = x$

$18.3 = x$



The guarantee should be for eighteen months.

Check

Let $x = 18$.

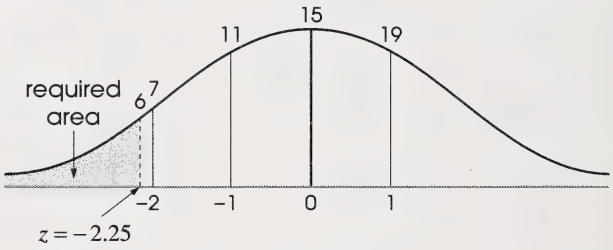
$$z = \frac{18 - 86}{36}$$
$$= -\frac{68}{36}$$
$$= -1.89$$

Area = 0.4706

required area = $0.5 - 0.4706$
$= 0.0294$
$\doteq 3\%$

A guarantee of eighteen months agrees with the percentage of machines that the manufacturer is willing to repair. **(2 marks)**

10. $$z = \frac{x - \mu}{\sigma}$$
$$= \frac{6 - 15}{4}$$
$$= -\frac{9}{4}$$
$$= -2.25$$

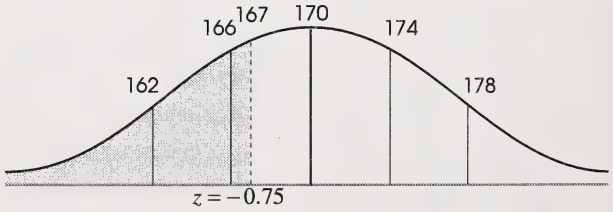


The area for $z = -2.25$ is 0.4878.

required area = $0.5 - 0.4878$
$= 0.0122$

The probability is 0.0122. **(2 marks)**

11. a. $$z = \frac{x - \mu}{\sigma}$$
$$= \frac{167 - 170}{4}$$
$$= -\frac{3}{4}$$
$$= -0.75$$



The area for $z = -0.75$ is 0.2734.

The required area is $0.5 - 0.2734 = 0.2266$.

$\therefore\ 0.2266 \times 200 = 45.32$

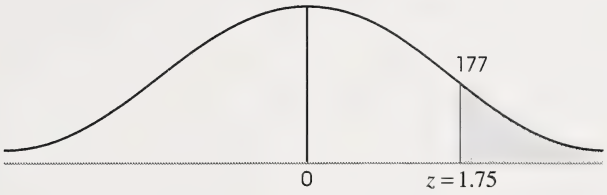
Thus, 45 students have heights of 167 cm or less. **(1 mark)**

b. $z = \dfrac{x-\mu}{\sigma}$

$= \dfrac{177-170}{4}$

$= \dfrac{7}{4}$

$= 1.75$



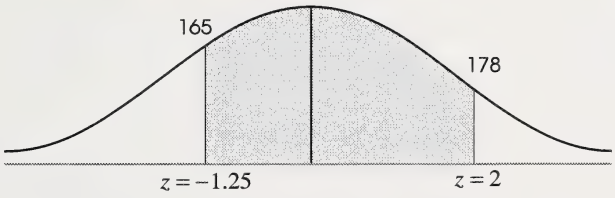
The area for $z = 1.75$ is 0.4599.

The required area is $0.5 - 0.4599 = 0.0401$.

$\therefore\ 0.0401 \times 200 = 8.02$

Thus, 8 students have heights of 177 cm or more. **(1 mark)**

c.



z-score for 165 cm

$z = \dfrac{165-170}{4}$

$= -\dfrac{5}{4}$

The area for $z = -1.25$ is 0.3944.

z-score for 178 cm

$z = \dfrac{178-170}{4}$

$= 2$

The area for $z = 2$ is 0.4772.

Total Area $= 0.3944 + 0.4772$

$= 0.8716$

$\therefore\ 0.8716 \times 200 = 174.32$

Thus, 174 students have heights between 165 cm and 178 cm. **(1 mark)**

d. To find 16% below a certain height, use $(50-16)$ 34%. From the tables, an area of 0.34 gives a z-score of -1.

$z = \dfrac{x-\mu}{\sigma}$

$-1 = \dfrac{x-170}{4}$ $(-0.995$ rounded to two decimal places is $-1.00.)$

$-4 = x - 170$

$x = 166$

Sixteen percent of the students are below 166 cm. **(1 mark)**

Final Module Assignment Answer Key (25 marks)

Part A: Multiple Choice

Each answer in this part is worth 1 mark.

1. B	**2.** A	**3.** D	**4.** B	**5.** B
6. C	**7.** D	**8.** B	**9.** A	**10.** B

(10 marks)

Part B: Numerical Response

Each answer in this part is worth 1 mark.

1. 0.01 **2.** 10 **3.** 1120 **4.** 35 **5.** 0.7

(5 marks)

Part C: Written Response

1.

$$
\begin{aligned}
{}_nP_2 &= 132 \\
\frac{n!}{(n-2)!} &= 132 \\
\frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} &= 132 \\
n(n-1) &= 132 \\
n^2 - n - 132 &= 0 \\
(n-12)(n+11) &= 0 \\
n - 12 &= 0 \\
n &= 12
\end{aligned}
$$

Jennifer has 12 different books in the set. **(2 marks)**

2. Any four programs: ${}_{15}C_4$

No academic program: ${}_5C_4$

One academic program: ${}_5C_3 \times {}_{10}C_1$

At least two academic programs:

$$
\begin{aligned}
{}_{15}C_4 - {}_5C_4 - \left({}_5C_3 \times {}_{10}C_1 \right) &= 1365 - 5 - 100 \\
&= 1260
\end{aligned}
$$

Sasha can choose four different programs in 1260 ways. **(3 marks)**

3.

$$
\begin{aligned}
{}_nC_2 - n &= 230 \\
\frac{n!}{(n-2)!2!} - n &= 230 \\
\frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}2} - n &= 230 \\
\frac{n(n-1)}{2} - n &= 230 \\
n^2 - n - 2n - 460 &= 0 \\
n^2 - 3n - 460 &= 0 \\
(n-23)(n+20) &= 0 \\
n - 23 &= 0 \\
n &= 23
\end{aligned}
$$

There are 23 sides. Carlos used $(n-20)=0$. He mixed up the signs of the factors for $n^2-3n-460$. **(3 marks)**

4. The president can be chosen in ${}_6C_1$ ways.

The secretary can be chosen in ${}_5C_1$ ways.

The treasurer can be chosen in ${}_4C_1$ ways.

number of committees $= {}_6C_1 \times {}_5C_1 \times {}_4C_1$

The position of the president is fixed; the position of the secretary and treasurer can be reversed (boy and girl or girl and boy).

$$
\begin{aligned}
\therefore \text{ number of committees} &= {}_6C_1 \times {}_5C_1 \times {}_4C_1 \times 2 \\
&= 6 \times 5 \times 4 \times 2 \\
&= 240
\end{aligned}
$$

A total of 240 committees can be selected. **(2 marks)**

Final Test

Included here is the answer key to the final test and the student's copy of the final test which is designed for photocopying and possible faxing.

Note

The answer key and student's copy of this final test should be kept secure by the teacher. Students should not have access to this test until it is assigned in a supervised situation. The answers should be stored securely and retained by the teacher at all times.

MATHEMATICS 30

FINAL TEST ANSWER KEY

Part A: Multiple Choice (40 marks)

Each answer in this part is worth 1 mark.

1.	D	**9.**	C	**17.**	A	**25.**	B	**33.**	D
2.	B	**10.**	D	**18.**	B	**26.**	D	**34.**	C
3.	D	**11.**	B	**19.**	A	**27.**	B	**35.**	A
4.	A	**12.**	C	**20.**	C	**28.**	B	**36.**	B
5.	A	**13.**	B	**21.**	D	**29.**	C	**37.**	D
6.	C	**14.**	A	**22.**	B	**30.**	D	**38.**	A
7.	A	**15.**	B	**23.**	C	**31.**	A	**39.**	D
8.	D	**16.**	C	**24.**	D	**32.**	B	**40.**	D

Part B: Numerical Response (8 marks)

Each answer in this part is worth 1 mark.

1.	0.5	**2.**	1.54	**3.**	35.8	**4.**	116
5.	1	**6.**	720	**7.**	1.33	**8.**	2

Part C: Written Response (22 marks)

1. The potential zeros of 6 are $\pm 1,\ \pm 2,\ \pm 3$, and ± 6.

-1	1	2	-5	-4	6
		-1	-3	2	6
	1	3	-2	-6	$\boxed{0}$

Since $P(-1)=0$, $x-1$ is a factor.

3	1	3	-2	-6
		3	0	-6
	1	0	-2	$\boxed{0}$

Since $P(3)=0$, $x+3$ is a factor.

$$Q(x)=x^2-2$$

Let $x^2-2=0$

$$x^2=2$$

$$x=\pm\sqrt{2}$$

When $x=0$, the y-intercept is 6.

The graph shows a y-intercept of 6. The four zeros of $P(x)$ are $-3,\ 1$, and $\pm\sqrt{2}$. The x-intercepts from the graph are $-3, 1$, a value between -2 and -1, and another value between 1 and 2. **(4 marks)**

2.

Parameter	Description	Effects on the Graph
$\frac{1}{2}$	amplitude	The distance between the maximum and minimum changed from 2 to 1.
$-\frac{\pi}{2}$	horizontal translation	moves the graph from $y = \cos\ \theta$, $\frac{\pi}{2}$ units to the right
2	vertical translation	moves the graph of $y = \cos\ \theta$, 2 units up

(4 marks)

3.

$$S_5 + t_6 = S_6$$
$$-610 + t_6 = 1820$$
$$t_6 = 2430$$

$$t_6 = ar^5 = 2430$$

$$t_4 = ar^3 = 270$$

$$\frac{ar^5}{ar^3} = \frac{2430}{270}$$
$$r^2 = 9$$
$$r = \pm 3$$

$$t_n = ar^{n-1}$$
$$t_4 = ar^3$$
$$270 = a(-3)^3$$
$$a = -10$$

$$S_n = \frac{a\left(r^n - 1\right)}{r - 1}$$
$$S_{10} = \frac{-10\left[(-3)^{10} - 1\right]}{-3 - 1}$$
$$= \frac{-10(59\,048)}{-4}$$
$$= 147\,620$$

The sum of the first ten terms is 147 620. **(5 marks)**

4. $t_7 = t_{6+1}$

$= {}_9C_6 (x)^{9-6} (-2y)^6$

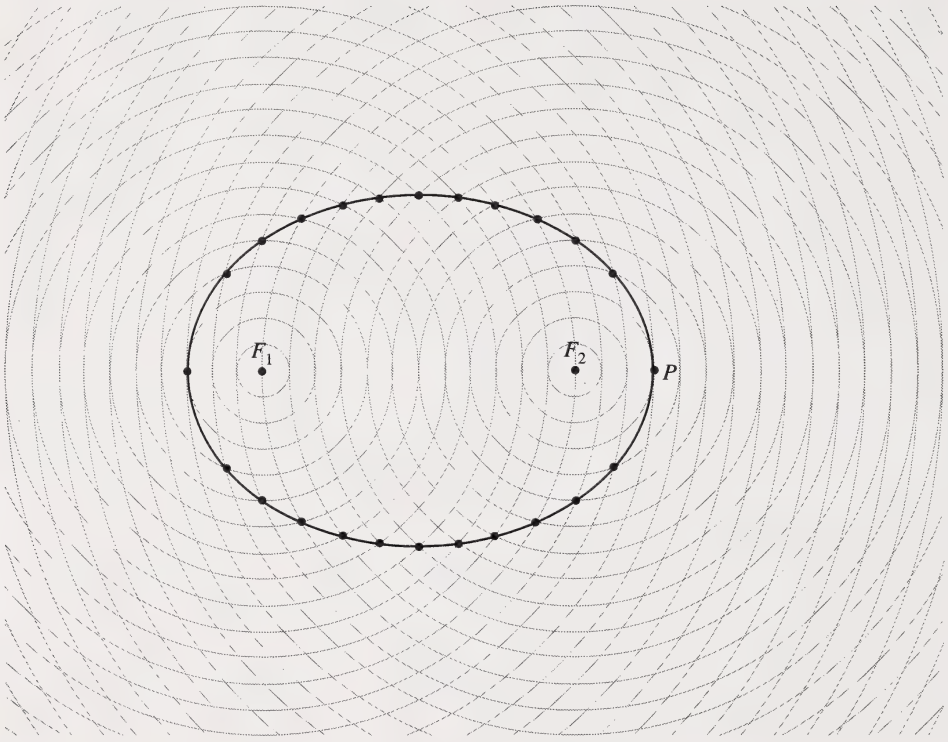
$= \frac{9!}{3!6!} x^3 \left(64y^6\right)$

$= \frac{9 \times 8 \times 7 \times \cancel{6!}}{3 \times 2 \times \cancel{6!}} x^3 \left(64y^6\right)$

$= 5376x^3y^6$

The seventh term is $5376x^3y^6$. **(3 marks)**

5. a.



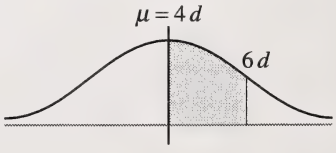
(1 mark)

b. The ellipse will change to a line segment. **(1 mark)**

c. The value of the eccentricity approaches zero. **(1 mark)**

6. To find the z-score, the area close to the mean must be calculated. Roshni did not realize the importance of this. The area between six days and four days (mean) is the difference of 0.5000 and 0.1844.

Therefore, Pierre's solution is correct. **(3 marks)**



MATHEMATICS 30

FINAL TEST

GENERAL INSTRUCTIONS

YOU HAVE **TWO AND ONE-HALF** $\left(2\frac{1}{2}\right)$ HOURS TO COMPLETE THIS TEST. Work through the entire test answering the questions you are sure you know. Do not spend too much time on any one problem. You will then be able to concentrate on the questions of which you are not quite sure.

TOTAL MARKS: 70

PART A: Multiple Choice	40 marks
PART B: Numerical Response	8 marks
PART C: Written Response	22 marks

Any hand-held calculator may be used. Calculators having graphing capabilities, built-in formulas, mathematical functions, or other programmable features are allowed.

All necessary tables and a list of formulas have been provided at the end of the test.

Value

40

PART A: MULTIPLE CHOICE

Answer all multiple-choice questions on the Part A: Response Page included in your test.

Read each question carefully and decide which of the choices BEST completes the statement or answers the question. Locate the question number on the Response Page and place your answer in the corresponding blank.

1. If $x^3 - 7x^2 + kx - 3$ is divided by $x - 4$ and the remainder is 18, then k is

A. -23
B. 5
C. 6
D. 15

2. If $P(3) = 16$ and $P(-2) = 16$ when $P(x) = ax^2 - 2bx - 8$, then the respective values of a and b are

A. 7 and -2
B. 4 and 2
C. 2 and 7
D. 2 and -5

3. The integral values at which the graph of $y = x^3 + 2x^2 - 7x + 4$ intersects the x-axis are

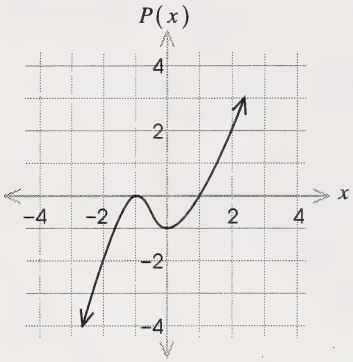
A. -2 and 4
B. -2 and -4
C. 2 and 4
D. 1 and -4

4. If the x-intercepts of the graph of a polynomial function are 2, $\sqrt{2}$, and $-\sqrt{2}$, then the polynomial could be

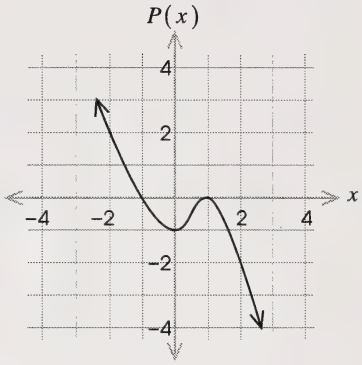
A. $x^3 - 2x^2 - 2x + 4$
B. $x^3 + 2x^2 - 2x - 4$
C. $x^3 + 2x^2 + 2x + 4$
D. $x^3 - 2\sqrt{2}x + 2\sqrt{2}$

5. If $P(x)=(x+1)^2(x-1)$, then the graph of $P(x)$ is

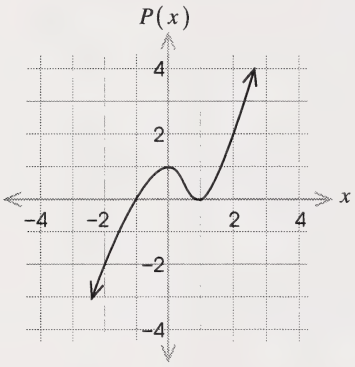
A.



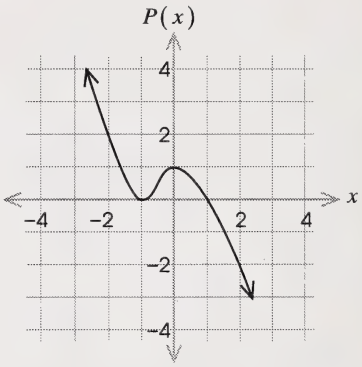
B.



C.



D.



6. When $6x^4-x^3+11x^2-2x-3$ is divided by $2x-1$, the quotient and remainder are

A. $6x^3-4x^2-13x+4.5$ and -3.25

B. $3x^3+x+6x+2$ and -0.5

C. $3x^3+x+6x+2$ and -1

D. $6x^3+2x^2+12x+4$ and 0

7. The solution set of $4\cos^2\theta=3$, where $0\le\theta\le2\pi$, is

A. $\left\{\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}\right\}$

B. $\left\{\frac{\pi}{6},\frac{\pi}{3},\frac{2\pi}{3},\frac{5\pi}{6}\right\}$

C. $\left\{\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6}\right\}$

D. $\left\{\frac{\pi}{12},\frac{5\pi}{12},\frac{7\pi}{12},\frac{11\pi}{12}\right\}$

8. If $\cos\theta$ is negative and $\sin\theta = \frac{3}{5}$, then $\tan\theta$ is

A. $\frac{3}{5}$

B. $\frac{5}{4}$

C. $-\frac{5}{3}$

D. $-\frac{3}{4}$

9. For the function $y = \frac{1}{3}\cos\frac{\theta}{4}$, the period is

A. 3π
B. 2π
C. 8π
D. 6π

10. An equivalent expression for $\frac{\cos^2\theta}{1+\sin\theta}$ is

A. $1+\sin\theta$
B. $\sin\theta$
C. $1+\cos\theta$
D. $1-\sin\theta$

11. The measure of an angle subtended at the centre of a circle by an arc equal in length to the radius of the circle is called the

A. arc length
B. radian
C. radius
D. central angle

12. The exact value of $\sin 135°$ is

A. $-\frac{\sqrt{2}}{2}$

B. $\frac{\sqrt{3}}{2}$

C. $\frac{\sqrt{2}}{2}$

D. $-\frac{\sqrt{3}}{3}$

13. From the top of a 120 m cliff, the angle of depression of a boat docked in the harbour below is 35°. How far is the boat from the base of the cliff? (Round the answer to the nearest metre.)

A. 236 m
B. 171 m
C. 84 m
D. 147 m

14. Cos 15° is equal to

A. $\frac{\sqrt{6}+\sqrt{2}}{4}$

B. $\frac{\sqrt{6}-\sqrt{2}}{4}$

C. $-\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)$

D. $\frac{1-\sqrt{3}}{8}$

15. An exponential form of $\log_M\left(\frac{1}{H}\right) = -R$ is

A. $H = M^{-R}$
B. $H = M^{R}$
C. $(-H)^{R} = \frac{1}{M}$
D. $\frac{1}{H} = -R_M$

16. Expressed as a single logarithm, $\log_4 x + y\log_4 z - w\log_4 v$ is

A. $\log_4 \frac{xy^z}{wv}$

B. $\log_4 \frac{x+z^y}{v^w}$

C. $\log_4 \frac{xz^y}{v^w}$

D. $\log_4 \frac{xz^y}{w^v}$

17. If $x = \log_3 243$, then x is equal to

A. 5
B. 3^{243}
C. 243^3
D. 81

18. The value of $\frac{1}{2}\log_2 256 - \log_5 625$ is

A. 4
B. 0
C. 3
D. 1

Use the following information to answer question 19.

$t_n = ar^{n-1}$, where t_n = the nth term
a = the first term
r = the common ratio
n = the number of a term

19. When the first term is 10, the common ratio is 3, and the nth term is 810, the logarithm expression for the number of terms n is

A. $\frac{\log 81 + \log 3}{\log 3}$
B. $\frac{\log 81 - 1}{\log 3}$
C. $\frac{\log 81 + \log 10}{\log 3}$
D. $\frac{\log 3}{\log 81} + 1$

20. If $\log_3 81 = 2x$ and $\log_5 125 = 3x - y$, then the value of y is

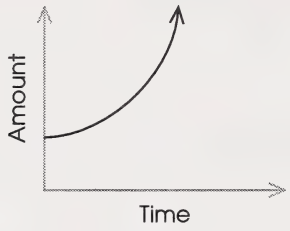
A. 2
B. 5
C. 3
D. 4

21. The graph of the function $y = a^x$ passes through the point $P(5, 12)$. Correct to the nearest whole number, the value of a is

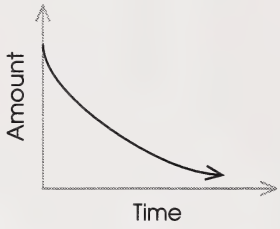
A. 35
B. 3
C. 1.6
D. 2

22. A radioactive substance decays exponentially. Half of its original amount remains after six hours. The graph that best represents this relation is

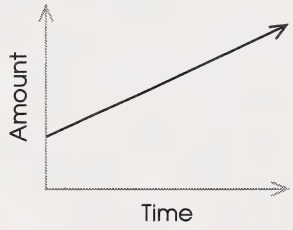
A.



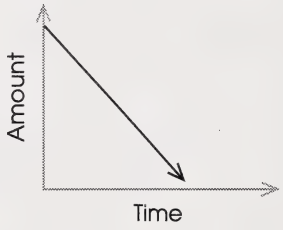
B.



C.



D.



23. In a geometric series if $t_5 = -324$ and $t_3 = -36$, then t_1 is

A. -2
B. 3
C. -4
D. -3

24. The number of positive integers between 24 and 362 that are divisible by 9 is

A. 46
B. 44
C. 45
D. 38

25. The geometric series with the common ratio $\sqrt{3}$ is

A. $3\sqrt{3}+4\sqrt{3}+5\sqrt{3}+\ldots$
B. $3+\sqrt{27}+9$
C. $3\sqrt{3}+6\sqrt{3}+9\sqrt{3}+\ldots$
D. $\sum_{n=1}^{5}\left(3+n\sqrt{3}\right)$

26. If the sum of the first n terms of an arithmetic series is n^2 and the first term is 1, then the nth term is

A. 2^{n-1}
B. 2^n-1
C. $2n+1$
D. $2n-1$

27. $\sum_{k=2}^{7} 3^{k-1}$ is equal to

A. 180
B. 1092
C. 360
D. 2196

28. A circle degenerates into

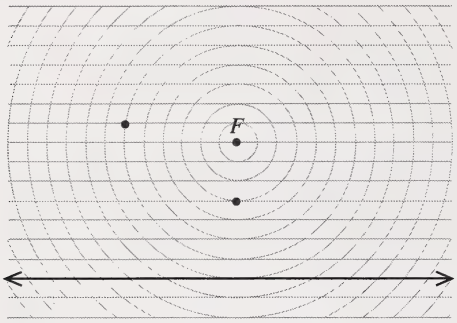
A. a straight line
B. a point
C. two adjacent points
D. two intersecting lines

29. A conic is described as having an eccentricity of 0.2. This conic is

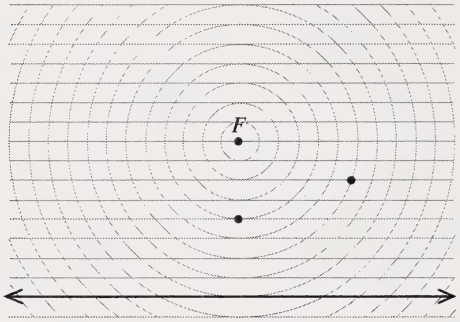
A. a circle
B. a parabola
C. an ellipse
D. a hyperbola

30. Which of the following graphs shows that two points lie on a parabola with a focus F, a directrix D, and an eccentricity of $\frac{4}{3}$?

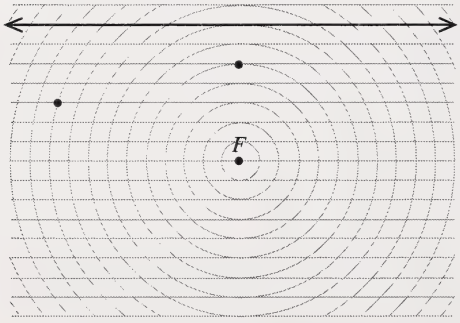
A.



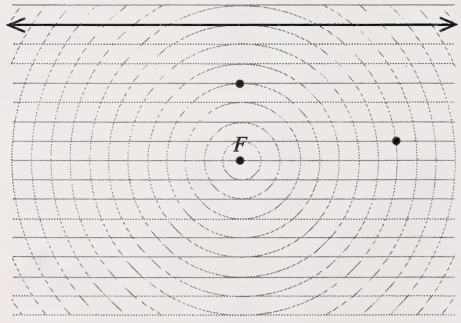
B.



C.



D.



31. A point (x, y) moves so that it is equidistant from a fixed point $(0, 4)$ and the line $y = 6$. A possible value of y when $x = 2$ is

A. 6
B. 4
C. 10
D. 2

32. A necessary condition for the quadratic relation $Ax^2 + Cy^2 + Dx + Ey + F = 0$ to be a hyperbola is

A. $AC > 0$ and $A \neq C$
B. $A > 0$, $C < 0$, and $A \neq C$
C. $AC < 0$ and $A \neq C$
D. $A = C$

33. The equation $5x^2 + 2y^2 - 8x - 2y - 3 = 0$. The value that would change the ellipse into a circle is

A. $D = 5$
B. $A = -8$
C. $E = 2$
D. $C = 5$

34. In how many ways can four different physics books and five different mathematics books be arranged on a shelf if the mathematics books are always kept together?

A. 604 800
B. 967 680
C. 14 400
D. 291 920

35. Five different flags can be used to send different signals by changing their positions on a flagpole. If at least three flags must be shown at any one time to represent one signal, then the number of different signals that can be shown is

A. 300
B. 181
C. 85
D. 185

36. A committee of four students is to be selected from six boys and four girls. How many different committees can be formed if, on any one committee, there must be no more than three girls and at least one girl?

A. 1220
B. 194
C. 80
D. 2880

37. There are eight men and six women in a dart club. If a group of three persons is chosen at random to form a committee, determine the probability, to two decimal places, that two of them are men.

A. 0.42
B. 0.34
C. 0.36
D. 0.46

38. The standard deviation (correct to two decimal places) of the following test scores is

68	86	74	44	98	52
60	79	48	58	74	65

A. 15.31
B. 13.66
C. 14.24
D. 15.45

39. The heights of 1600 students are analysed. If the heights are distributed about a mean of 174 cm with a standard deviation of 12.5 cm, then the expected number of students who are shorter than 148.5 cm is

A. 192
B. 1008
C. 777
D. 33

40. For a normal distribution the mean (μ) is 16 and the standard deviation (σ) is 6. The probability of x, to two decimal places, for the interval $12.8 \leq x \leq 17.4$ is

A. 0.09
B. 0.23
C. 0.53
D. 0.29

PART A: RESPONSE PAGE

_____	1.	_____	16.	_____	31.
_____	2.	_____	17.	_____	32.
_____	3.	_____	18.	_____	33.
_____	4.	_____	19.	_____	34.
_____	5.	_____	20.	_____	35.
_____	6.	_____	21.	_____	36.
_____	7.	_____	22.	_____	37.
_____	8.	_____	23.	_____	38.
_____	9.	_____	24.	_____	39.
_____	10.	_____	25.	_____	40.
_____	11.	_____	26.		
_____	12.	_____	27.		
_____	13.	_____	28.		
_____	14.	_____	29.		
_____	15.	_____	30.		

Name of Student _____ Student I.D. # _____

Name of School _____ Date _____

Value

PART B: NUMERICAL RESPONSE

8

Instructions

- Consider all numbers used in the questions to be **exact positive real** numbers and not the result of a measurement.
- **Enter the first digit of your answer in the left-hand box and leave any unused boxes blank.**
- If you wish to change an answer, erase **all** traces of your first answer.

Sample Questions and Solutions

Correct to the nearest tenth, $\sin 44°$ is equal to _____.

$\sin 44° = 0.6947$

Record 0.7. ⟶

In the equation $85 = -8 + (n-1)(3)$, the value of n is _____.

$85 = -8 + (n-1)(3)$

$93 = 3n - 3$

$n = 32$

Record 32. ⟶

1. One x-intercept of the graph of $y = 2x^2 + mx - 3$ is -3. The other x-intercept is _____.

2. If $\log_5 12 = x$, then the value of x, correct to the nearest hundredth, is _____.

3. A 40 cm pendulum swings through an arc of 25 cm. The angle through which the pendulum swings, correct to the nearest tenth, is _____.

Name of Student _____ Student I.D. # _____

Name of School _____ Date _____

4. Jackie begins a savings plan by saving \$4.00 during the first week. In each subsequent week she saves three dollars more than the week before. At the end of the eighteenth week, the total amount she has saved is \$ _____.

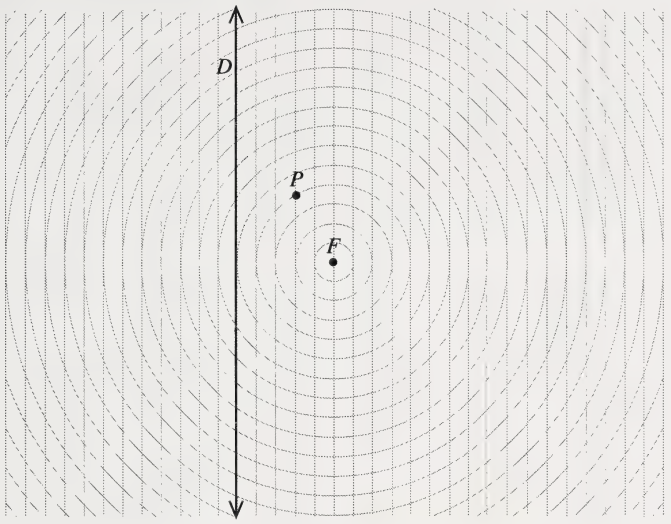
5. The number of bacteria in a test tube is determined by the function $A = A_0 (2)^{\frac{t}{d}}$, where A_0 is the initial number of bacteria and d is the doubling time. A biologist began a culture with 50 bacteria and estimates that there are 3000 bacteria after 8 hours. The doubling time to the nearest hour is _____.

6. Eight people are seated at a round table. If three people must always sit together, then the number of possible arrangements is _____.

Name of Student _____ Student I.D. # _____

Name of School _____ Date _____

7. P is a point on a curve, F is the focus, and D is the directrix of the graph shown. The eccentricity, to the nearest hundredth, is _____.



8. The Bright-Lite bulb manufacturer has determined that the lives of these light bulbs are normally distributed with a mean of 5000 h and a standard deviation of 1000 h. If the manufacturer guarantees these bulbs for 3000 h, the percentage, to the nearest whole number, they expect to replace is _____.

Name of Student _____ Student I.D. # _____

Name of School _____ Date _____

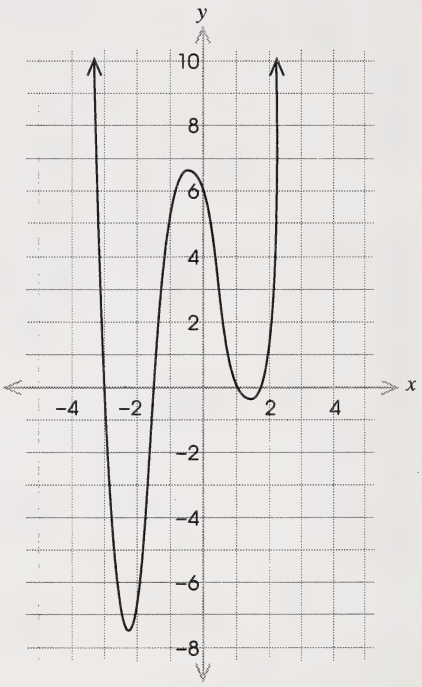
Value

PART C: WRITTEN RESPONSE

22 Answer each of the following questions using the space provided. Show all work. Part marks will be given for partially correct solutions.

(4 marks) **1.** The graph of $P(x) = x^4 + 2x^3 - 5x^2 - 4x + 6$ is shown here.

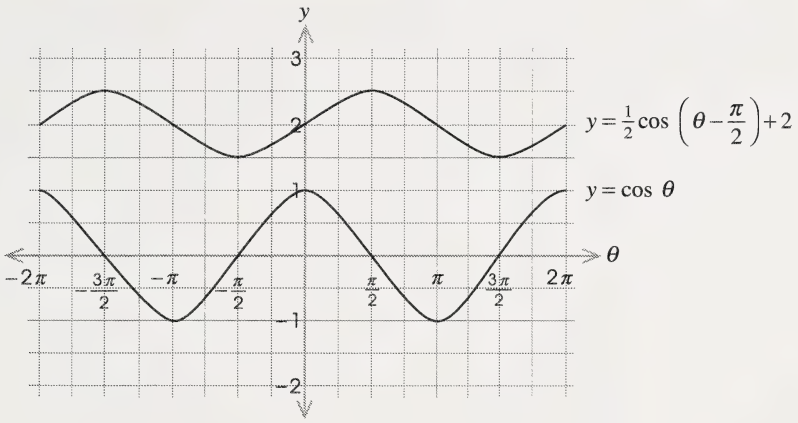
Verify algebraically that the graph shown is the graph of $P(x)$. Justify your verification.



Name of Student _____ Student I.D. # _____

Name of School _____ Date _____

(4 marks) **2.** The graphs of $y = \cos\ \theta$ and $y = \frac{1}{2}\cos\left(\theta - \frac{\pi}{2}\right) + 2$ are shown. Describe the effects of the parameters $\frac{1}{2}$, $-\frac{\pi}{2}$, and 2 on the graph of $y = \cos\ \theta$.



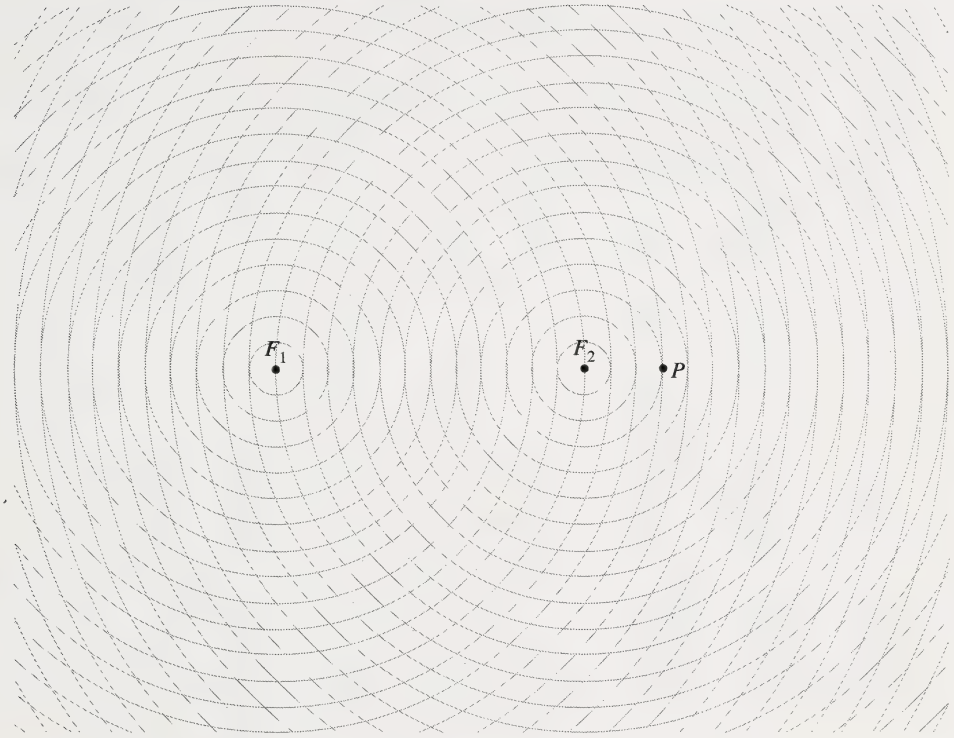
Name of Student _____	Student I.D. # _____
Name of School _____	Date _____

(5 marks) **3.** In a geometric series the sum of the first five terms is -610 and the sum of the first six terms is 1820. The fourth term is 270. Calculate the sum of the first ten terms for the series.

(3 marks) **4.** Determine the seventh term in the expansion of $(x-2y)^9$.

Name of Student _____	Student I.D. # _____
Name of School _____	Date _____

(1 mark) **5. a.** F_1 and F_2, the focal points of a quadratic relation, and a point P on that quadratic relation are shown. Complete the graph of the quadratic relation by plotting the points.



(1 mark) **b.** Explain what happens to the quadratic relation if the focal points are moved away from each other.

_____

_____

(1 mark) **c.** Explain what happens to the value of the eccentricity of this quadratic relation if the foci are moved close together.

_____

_____

Name of Student _____ Student I.D. # _____

Name of School _____ Date _____

(3 marks) **6.** Quality Meats keeps packaged ham on the shelf for an average of four days. The probability of selecting a package that is on the shelf for more than six days is 0.1844.

Two students, Roshni and Pierre, had to find the standard for the shelf period of the ham. Their solutions are given here.

Roshni's solution

z-score for $0.1844 = 0.48$

$$z = \frac{x - \mu}{\sigma}$$

$$0.48 = \frac{6 - 4}{\sigma}$$

$$8 = \frac{2}{0.48}$$

$$\doteq 4.2$$

Pierre's solution

Required area $\doteq 0.5000 - 0.1844$

$$\doteq 0.3156$$

z-score for $0.3156 = 0.9$

$$z = \frac{x - \mu}{\sigma}$$

$$0.9 = \frac{6 - 4}{\sigma}$$

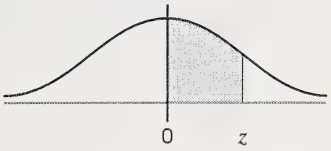
$$\sigma = \frac{2}{0.9}$$

$$\doteq 2.2$$

Determine which solution, Roshni's or Pierre's, is correct. Explain how you made your decision.

Name of Student _____	Student I.D. # _____
Name of School _____	Date _____

STANDARD NORMAL DISTRIBUTION TABLE



z	0	1	2	3	4	5	6	7	8	9
0.0	0.0000	0.0040	00080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0754
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0. 1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2258	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2518	0.2549
0.7	0.2580	0.2612	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2996	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4995	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993
3.2	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995
3.3	0.4995	0.4995	0.4995	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998
3.5	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998
3.6	0.4998	0.4998	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.7	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.8	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.9	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000

MATHEMATICS 30 FORMULA SHEET

The following information may be useful in writing this examination.

I. Polynomial Functions

- $P(x) = D(x)Q(x) + R$
- The roots of a quadratic equation are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

II. Trigonometry

- arc length $a = r\theta$
- $\sec A = \dfrac{1}{\cos A}$
- $\csc A = \dfrac{1}{\sin A}$
- $\cot A = \dfrac{\cos A}{\sin A}$
- $\sin^2 A + \cos^2 A = 1$
- $1 + \tan^2 A = \sec^2 A$
- $1 + \cot^2 A = \csc^2 A$
- $\sin(A + B) = \sin A \cos B + \cos A \sin B$
- $\cos(A - B) = \cos A \cos B + \sin A \sin B$
- $\sin(A - B) = \sin A \cos B - \cos A \sin B$
- $\cos(A + B) = \cos A \cos B - \sin A \sin B$

III. Statistics

- $z = \dfrac{x - \mu}{\sigma}$
- $y = mx + b$

IV. Quadratic Relations

- $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$
- $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- eccentricity $e = \dfrac{|\overline{PF}|}{|\overline{PD}|}$, where F = focus, D = directrix, and P = point on the conic

V. Permutations and Combinations

- ${}_nC_r = \dfrac{n!}{r!(n-r)!}$
- ${}_nP_r = \dfrac{n!}{(n-r)!}$
- $n! = n(n-1)(n-2)\ldots(3)(2)(1)$
- $(x+y)^n = {}_nC_0 x^n + {}_nC_1 x^{n-1} y + {}_nC_2 x^{n-2} y^2 + \ldots + {}_nC_k x^{n-k} y^k + \ldots + {}_nC_n y^n$

  General Term: $t_{k+1} = {}_nC_k x^{n-k} y^k$

VI. Sequences and Series

- $t_n = a + (n-1)d$
- $S_n = \dfrac{n(a + t_n)}{2}$
- $S_n = \dfrac{n[2a + (n-1)d]}{2}$
- $S_n = \dfrac{a(r^n - 1)}{r - 1}, \; r \neq 1$
- $t_n = ar^{n-1}$

TEACHER QUESTIONNAIRE FOR MATHEMATICS 30

This is a course designed in a new distance-learning format, so we are interested in your responses. Your constructive comments will be greatly appreciated so that a future revision may incorporate any necessary improvements.

Teacher's Name _____ Area of Expertise _____

School Name _____ Date _____

Design

1. The modules follow a definite systematic design. Did you find it easy to follow?

   ❑ Yes ❑ No If no, explain.

   _____

   _____

2. Did your observations reveal that the students found the design easy to follow?

   ❑ Yes ❑ No If no, explain.

   _____

   _____

3. Did you find the Learning Facilitator's Manual helpful?

   ❑ Yes ❑ No If no, explain.

   _____

   _____

4. Part of the design involves stating the objectives in student terms. Do you feel this helped the students understand what they were going to learn?

   ❑ Yes ❑ No If no, explain.

   _____

   _____

5. The Learning Facilitator's Manual contains Assignment answers and a sample test. Did you find these helpful?

❑ Yes ❑ No If no, explain.

_____

_____

6. Did the Follow-up Activities prove to be helpful?

❑ Yes ❑ No If no, explain.

_____

_____

7. Were students motivated to try these Follow-up Activities?

❑ Yes ❑ No If no, give details.

_____

_____

8. Suggestions for computer and video activities are included in the course. Were your students able to use these activities?

❑ Yes ❑ No Comment on the lines below.

_____

_____

9. Were the assignments appropriate?

❑ Yes ❑ No If no, give details.

_____

_____

10. Did you fax assignments? ❑ Yes ❑ No

11. If you did fax, did you get satisfactory results from using this procedure?

❑ Yes ❑ No If no, give details.

_____

_____

Instruction

1. Did you find the instruction clear?

   ❑ Yes ❑ No If no, give details.

   _____

   _____

2. Did your observations reveal that the students found the instruction interesting?

   ❑ Yes ❑ No If no, give details.

   _____

   _____

3. Did you find the instruction adequate?

   ❑ Yes ❑ No If no, give details.

   _____

   _____

4. Was the reading level appropriate?

   ❑ Yes ❑ No If no, give details.

   _____

   _____

5. Was the work load adequate?

   ❑ Yes ❑ No If no, give details.

   _____

   _____

6. Was the content accurate and current?

   ❑ Yes ❑ No If no, give details.

   _____

   _____

7. Did the content flow consistently and logically?

❑ Yes ❑ No If no, give details.

_____

_____

8. Was the transition between booklets smooth?

❑ Yes ❑ No If no, give details.

_____

_____

9. If applicable, was the transition between print and other media smooth?

❑ Yes ❑ No If no, give details.

_____

_____

Additional Comments

_____

_____

_____

_____

_____

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_____

Thanks for taking the time to complete this survey. Your feedback is important to us.

Fax Number: 674-6686

Instructional Design and Development
Alberta Distance Learning Centre
Box 4000
Barrhead, Alberta
T7N 1P4

Note: Please ensure that each of your students has completed and forwarded a copy of the Course Survey.

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Mathematics 30
Learning Facilitator's Manual

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